Algebra Concepts

An algebra foundations companion with concise notes, worked examples, and quick references for class, study, and review.

Module 1: Linear Equations, Inequalities, and Functions

Solve and model with linear equations and inequalities, graph lines, and build the function language needed for nonlinear and piecewise functions.

Mod 1 A Solving Linear Equations

Solve linear equations from one-step forms through equations with distribution, fractions, and special solution cases.

QUICK REFERENCE

- Use inverse operations to isolate the variable while keeping both sides balanced.
- Distribute and combine like terms before isolating the variable.
- Clear fractions by multiplying every term by a common denominator.
- A true statement after the variable disappears means all real numbers; a false statement means no solution.
- Check a proposed solution in the original equation.

Example 1. Solve $-3y + 12 = -3(2y + 5)$.

Answer: $y = -9$.

Example 2. Solve $\frac{5x}{6} - 7 = \frac{x}{3} + 3$.

Answer: $x = 20$.

Example 3. Solve $-5(4x - 5) = 5x$.

Answer: $x = 1$.

Example 4. Solve $4(2n - 3) + 5 = 3(n + 1)$.

Answer: $n = 2$.

Example 5. Solve and classify $2(3x + 4) = 6x + 8$.

Answer: All real numbers.

Example 6. Solve and classify $3(x - 2) = 3x + 5$.

Answer: No solution, $\varnothing$.

Mod 1 B Translating Equations and Word Problems

Translate verbal statements into equations and use those equations to solve number and applied problems.

QUICK REFERENCE

- Choose a variable and state what it represents before writing the equation.
- Words such as sum, difference, product, quotient, more than, and less than signal operations.
- "Less than" reverses the written order: five less than x is $x - 5$.
- Translate the relationship first, solve second, and answer the question with units.
- Check whether the result is reasonable in the original context.

Example 1. The sum of twice a number and 7 equals the sum of the number and 19. Write an equation and find the number.

Answer: $2x + 7 = x + 19$, **so the number is** 12.

Example 2. Four times a number minus 9 equals three times the number plus 4. Find the number.

Answer: $4x - 9 = 3x + 4$, **so** $x = 13$.

Example 3. Twice the difference of a number and 5 is 18. Find the number.

Answer: $2(x - 5) = 18$, **so** $x = 14$.

Example 4. Two consecutive integers have a sum of 51. Find them.

Answer: $x + (x + 1) = 51$. **The integers are** 25 **and** 26.

Example 5. A streaming service charges $\$12$ plus $\$3$ per movie. A bill is $\$33$. How many movies were rented?

Answer: $12 + 3m = 33$, **so** 7 **movies were rented.**

Mod 1 C Equations With More Than One Variable

Rearrange formulas for a requested variable and use common formulas in applications.

QUICK REFERENCE

- A literal equation contains two or more variables. Solve for the requested variable as if the other variables were known numbers.
- Undo addition and subtraction before multiplication and division when isolating a variable.
- Factor out the requested variable when it appears in more than one term.
- Common applications use $A = lw$, $d = rt$, and perimeter formulas.
- Keep units attached when substituting into a formula.

Example 1. Solve $V = APK$ for A.

Answer: $A = \dfrac{V}{PK}$.

Example 2. Solve $2x + y = 7$ for y.

Answer: $y = 7 - 2x$.

Example 3. Solve $P = a + b + c$ for c.

Answer: $c = P - a - b$.

Example 4. Solve $y = mx + b$ for x, assuming $m \neq 0$.

Answer: $x = \dfrac{y - b}{m}$.

Example 5. A rectangular sign has area 3763 square feet and width 53 feet. Find its height.

Answer: Using $A = wh$, $h = \dfrac{3763}{53} = 71$ feet.

Example 6. A ferry travels at 59 miles per hour for $3\dfrac{1}{2}$ hours. How far does it travel?

Answer: $d = rt = 59(3.5) = 206.5$ miles.

Mod 1 D Absolute Value Equations

Solve absolute value equations by separating distance statements into cases.

QUICK REFERENCE

- Absolute value measures distance from zero, so $|u| = k$ with $k > 0$ gives $u = k$ or $u = -k$.
- Isolate the absolute value expression before writing the two cases.
- If $|u| = 0$, then $u = 0$ gives one solution.
- An absolute value cannot equal a negative number, so $|u| = -k$ has no solution when $k > 0$.
- Check both candidates in the original equation.

Example 1. Solve $|2x - 11| = 17$.

Answer: $x = 14$ or $x = -3$.

Example 2. Solve $\left|\dfrac{x}{4} + 9\right| = 9$.

Answer: $x = 0$ or $x = -72$.

Example 3. Solve $|8z| = 0$.

Answer: $z = 0$.

Example 4. Solve $3|x + 2| - 4 = 17$.

Answer: $x = 5$ or $x = -9$.

Example 5. Solve $5 - |2y - 1| = 9$.

Answer: No solution, $\varnothing$.

Mod 1 E Solving Linear Inequalities

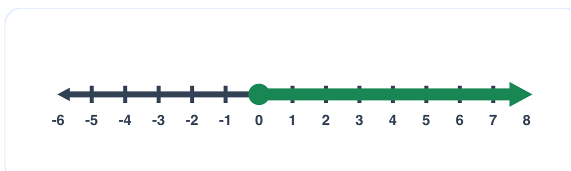
Solve linear inequalities and express solutions with inequality, interval, and number-line notation.

QUICK REFERENCE

- Solve a linear inequality like an equation, but reverse the inequality sign when multiplying or dividing by a negative number.
- Use an open endpoint for $<$ or $>$, and a closed endpoint for $\leq$ or $\geq$.
- Parentheses mark excluded interval endpoints; brackets mark included endpoints.
- Infinity always uses a parenthesis.
- Translate verbal conditions into inequalities before solving, and keep units attached in applications.

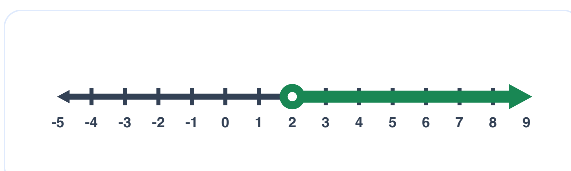
Example 1. Solve $x - 4 \geq -4$. Give interval notation and a number-line graph.

Answer: $x \geq 0$, or $[0, \infty)$.



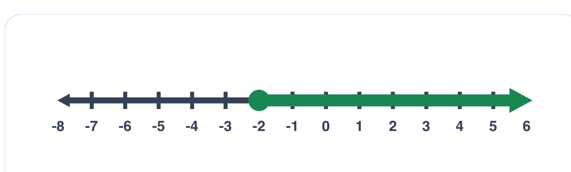
Example 2. Solve $5x - 6 > 4x - 4$. Give interval notation and a number-line graph.

Answer: $x > 2$, or $(2, \infty)$.



Example 3. Solve $-3(2x - 1) \leq 15$.

Answer: $x \geq -2$, or $[-2, \infty)$.



Example 4. Solve $\dfrac{x-1}{3} < \dfrac{x+5}{2}$.

Answer: $x > -17$, or $(-17, \infty)$.

Example 5. Seven more than twice a number is less than -13. Find all numbers that make the statement true.

Answer: $2x + 7 < -13$, so $x < -10$, or $(-\infty, -10)$.

Example 6. A rectangle has perimeter no greater than 80 centimeters and width 15 centimeters. Find its maximum length.

Answer: $2L + 2(15) \leq 80$, so $L \leq 25$. **The maximum length is 25 centimeters.**

Mod 1 F Compound Inequalities

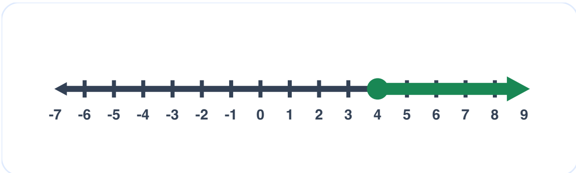
Solve intersections joined by "and" and unions joined by "or."

QUICK REFERENCE

- An "and" compound inequality requires both conditions, so keep the overlap of the solution sets.
- An "or" compound inequality accepts either condition, so combine both solution sets.
- A three-part inequality can be solved by performing the same operation on all three parts.
- Use $\cup$ to join separated intervals.
- Some "and" statements have no overlap; some "or" statements cover all real numbers.

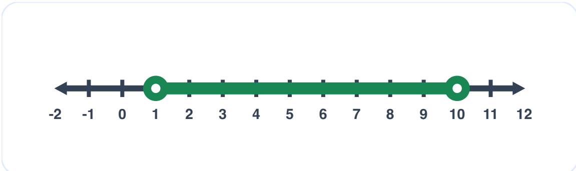
Example 1. Solve $x + 10 \geq 5$ and $2x - 4 \geq 4$.

Answer: $x \geq 4$, or $[4, \infty)$.



Example 2. Solve $-3x < -3$ and $x - 4 < 6$.

Answer: $1 < x < 10$, or $(1, 10)$.

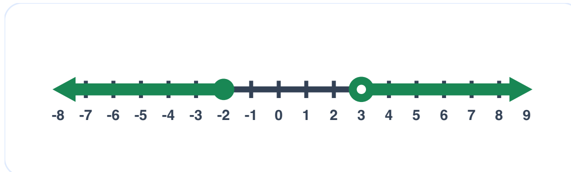


Example 3. Solve $-7 \leq 2x + 1 < 9$.

Answer: $-4 \leq x < 4$, or $[-4, 4)$.

Example 4. Solve $3x - 2 \leq -8$ or $2x + 5 > 11$.

Answer: $x \leq -2$ or $x > 3$, or $(-\infty, -2] \cup (3, \infty)$.



Example 5. Solve $x > 5$ and $x \leq 2$.

Answer: No solution, $\varnothing$.

Mod 1 G Absolute Value Inequalities

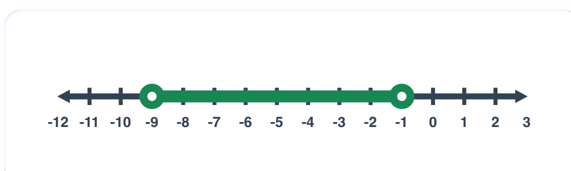
Translate absolute value inequalities into compound inequalities and graph their solution sets.

QUICK REFERENCE

- For $|u| < k$, values lie inside: $-k < u < k$. The same inside pattern works with $\leq$.
- For $|u| > k$, values lie outside: $u < -k$ or $u > k$. The same outside pattern works with $\geq$.
- Isolate the absolute value before applying the inside or outside pattern.
- When $k < 0$, compare the nonnegative absolute value with a negative number before doing algebra.
- Match strict inequalities with open endpoints and inclusive inequalities with closed endpoints.

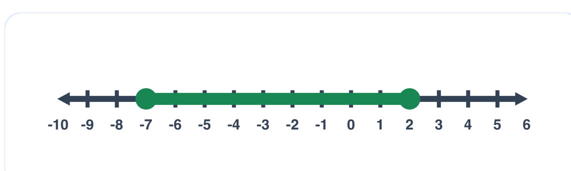
Example 1. Solve $|x + 5| < 4$.

Answer: $-9 < x < -1$, or $(-9, -1)$.



Example 2. Solve $|2x + 5| \leq 9$.

Answer: $-7 \leq x \leq 2$, or $[-7, 2]$.



Example 3. Solve $|3x - 1| > 8$.

Answer: $x < -\dfrac{7}{3}$ or $x > 3$.

Example 4. Solve $2|x-4|+1 \geq 11$.

Answer: $x \leq -1$ or $x \geq 9$, or $(-\infty, -1] \cup [9, \infty)$.

Example 5. Solve $|x-2| < -3$.

Answer: No solution, $\varnothing$.

Example 6. Solve $|x-2| \geq -3$.

Answer: All real numbers, $(-\infty, \infty)$.

Mod 1 H Graphing Linear Equations by Plotting Points

Create ordered-pair tables and graph linear equations from calculated points.

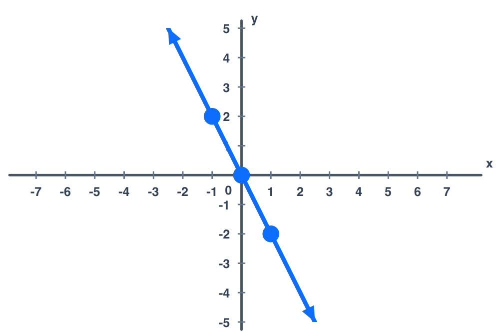
QUICK REFERENCE

- Choose convenient input values, substitute them into the equation, and record the ordered pairs.
- Two distinct points determine a line; a third point helps check the work.
- Plot coordinates as (x, y), then draw one straight line through the points.
- For $y = mx + b$, the y-intercept $(0, b)$ is often a convenient first point.
- For equations not solved for y, choose x-values that make the corresponding y-values easy to calculate.

Example 1. Complete a table for $y = -2x$ at $x = -1, 0, 1$, then graph the line.

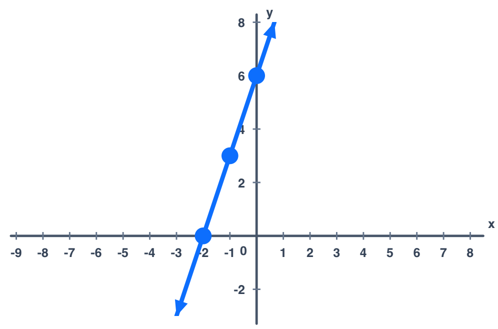
Answer: The points are $(-1, 2), (0, 0), (1, -2)$.

x	-1	0	1
y	2	0	-2



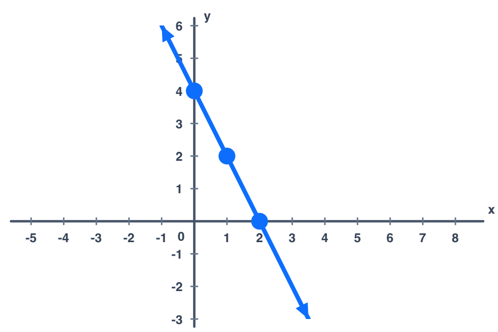
Example 2. Complete a table for $y = 3x + 6$ at $x = -2, -1, 0$, then graph the line.

Answer: The points are $(-2, 0), (-1, 3), (0, 6)$.



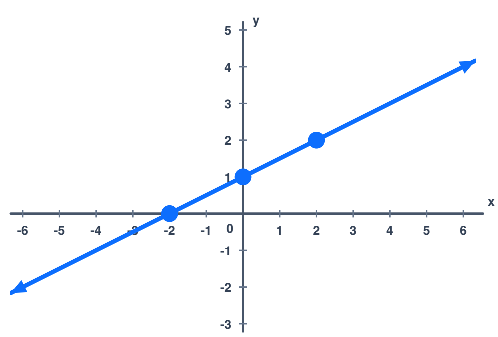
Example 3. Find three points on $2x + y = 4$ and graph the line.

Answer: One convenient set is $(0, 4), (1, 2), (2, 0)$.



Example 4. Graph $y = \frac{1}{2}x + 1$ by plotting three points.

Answer: One convenient set is $(-2, 0), (0, 1), (2, 2)$.



Mod 1 I Graphing Linear Equations by Plotting Intercepts

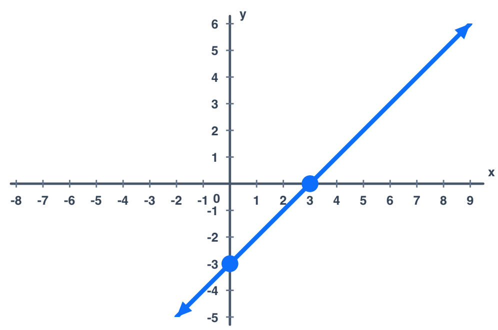
Find x- and y-intercepts and use them to graph linear equations.

QUICK REFERENCE

- To find the x-intercept, set $y = 0$ and solve for x.
- To find the y-intercept, set $x = 0$ and solve for y.
- Plot both intercepts and draw the line through them.
- A line through the origin has the same x- and y-intercept, so find another point.
- Horizontal and vertical lines may have only one type of intercept.

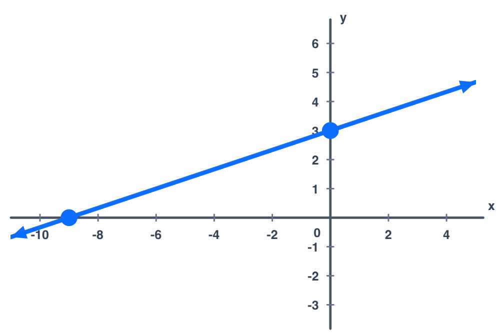
Example 1. Find the intercepts and graph $x - y = 3$.

Answer: **x-intercept:** $(3, 0)$. **y-intercept:** $(0, -3)$.



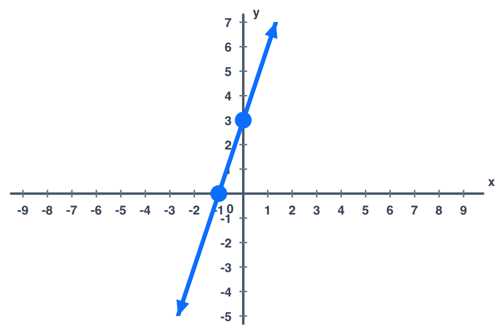
Example 2. Find the intercepts and graph $-x + 3y = 9$.

Answer: **x-intercept:** $(-9, 0)$. **y-intercept:** $(0, 3)$.



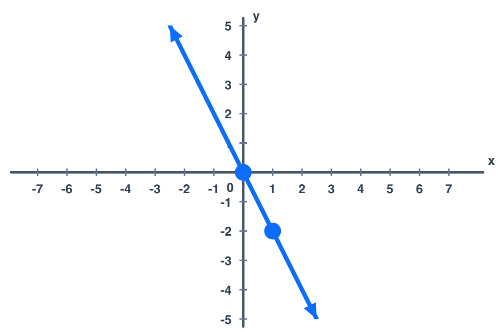
Example 3. Find the intercepts and graph $6x - 2y = -6$.

Answer: **x-intercept:** $(-1, 0)$. **y-intercept:** $(0, 3)$.



Example 4. Graph $4x + 2y = 0$ using its intercept and one additional point.

Answer: **Both intercepts are** $(0, 0)$. **Another point is** $(1, -2)$.



Mod 1 J Slope

Find slope from ordered pairs and linear equations.

QUICK REFERENCE

- Slope is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, or rise divided by run.
- A horizontal line has slope 0.
- A vertical line has undefined slope because its run is 0.
- In $y = mx + b$, the coefficient m is the slope.
- For an equation in standard form, solve for y to identify the slope.

Example 1. Find the slope through $(10, -5)$ and $(8, 5)$.

Answer: $m = \dfrac{5 - (-5)}{8 - 10} = -5$.

Example 2. Find the slope through $(-8, 1)$ and $(-8, 10)$.

Answer: The slope is undefined.

Example 3. Find the slope through $(-3, 6)$ and $(4, 6)$.

Answer: $m = 0$.

Example 4. Find the slope of $y = 4x + 7$.

Answer: The slope is 4.

Example 5. Find the slope of $3x - 2y = 6$.

Answer: Solving for y **gives** $y = \frac{3}{2}x - 3$**, so the slope is** $\frac{3}{2}$.

Mod 1 K Writing Equations of Lines

Write slope-intercept equations from slopes, points, and intercepts.

QUICK REFERENCE

- Slope-intercept form is $y = mx + b$.
- Point-slope form is $y - y_1 = m(x - x_1)$.
- Find slope first when two points are given, then substitute one point.
- If the y-intercept is given, place it directly into $y = mx + b$.
- Simplify coefficients and constants before writing the final equation.

Example 1. Write the line with slope $-\frac{2}{9}$ through $(0, 8)$ in slope-intercept form.

Answer: $y = -\frac{2}{9}x + 8$.

Example 2. Write the line through $(6, 3)$ and $(3, 6)$.

Answer: $y = -x + 9$.

Example 3. Write the line with slope 3 through $(-4, 12)$.

Answer: $y = 3x + 24$.

Example 4. Write the line with slope -4 and y-intercept $(0, 4)$.

Answer: $y = -4x + 4$.

Example 5. Write the line through $(0, 0)$ and $(4, 7)$.

Answer: $y = \frac{7}{4}x$.

Example 6. Write the line with slope $-\frac{5}{6}$ through $(-1, -7)$.

Answer: $y = -\frac{5}{6}x - \frac{47}{6}$.

Mod 1 L Functions, Domain, and Range

Identify functions and determine domain and range from relations and equations.

QUICK REFERENCE

- A relation is a function when every input has exactly one output.
- Domain is the set of inputs; range is the set of outputs. Do not repeat values in a set.
- Repeated outputs are allowed, but a repeated input cannot have different outputs.
- An equation describes a function when each x-value determines exactly one y-value.
- A vertical equation such as $x = c$ does not describe y as a function of x.

Example 1. Find the domain and range of $\{(8, 7), (1, 5), (-8, 9), (9, -8)\}$.

Answer: Domain: $\{-8, 1, 8, 9\}$. **Range:** $\{-8, 5, 7, 9\}$.

Example 2. Find the domain and range of $\{(10, 5), (5, 5), (2, 5)\}$.

Answer: Domain: $\{2, 5, 10\}$. **Range:** $\{5\}$.

Example 3. Is $\{(7, -8), (-2, -5), (8, 8), (-3, 9)\}$ a function?

Answer: Yes. Every input occurs only once.

Example 4. Is $\{(1, 4), (2, 6), (1, 9)\}$ a function?

Answer: No. Input 1 has two outputs.

Example 5. Does $y = -2x - 4$ describe a function?

Answer: Yes. Every x-value gives exactly one y-value.

Example 6. Does $x = -4$ describe y as a function of x?

Answer: No. The single input $x = -4$ is paired with many y-values.

Mod 1 M Function Notation

Evaluate functions at specified inputs using function notation.

QUICK REFERENCE

- $f(a)$ means substitute a for every x in the rule.
- Function notation does not mean multiplication.
- Use parentheses carefully when substituting negative numbers.

- Evaluate each requested input separately and simplify the resulting number.
- The same substitution process works for linear, quadratic, and other polynomial rules.

Example 1. For $f(x) = -5x - 5$, find $f(-1)$, $f(0)$, and $f(3)$.

Answer: $f(-1) = 0$, $f(0) = -5$, and $f(3) = -20$.

Example 2. For $g(x) = x^2 + 5$, find $g(-3)$, $g(0)$, and $g(3)$.

Answer: $g(-3) = 14$, $g(0) = 5$, and $g(3) = 14$.

Example 3. For $h(x) = 2x^3$, find $h(-2)$, $h(0)$, and $h(3)$.

Answer: $h(-2) = -16$, $h(0) = 0$, and $h(3) = 54$.

Example 4. For $r(x) = -7x$, find $r(-2)$, $r(0)$, and $r(2)$.

Answer: $r(-2) = 14$, $r(0) = 0$, and $r(2) = -14$.

Example 5. For $q(x) = 6x^2 + 5$, find $q(-3)$, $q(0)$, and $q(1)$.

Answer: $q(-3) = 59$, $q(0) = 5$, and $q(1) = 11$.

Mod 1 N Equations of Lines in Function Notation

Write linear equations using function notation.

QUICK REFERENCE

- Replace y with $f(x)$ to write a line as a function.
- A line with slope m and y-intercept $(0, b)$ is $f(x) = mx + b$.
- Use point-slope form to find the rule when a slope and non-intercept point are given.
- When two points are given, calculate the slope before finding the y-intercept.
- A slope of 0 produces a constant function $f(x) = b$.

Example 1. Write the line with slope -2 and y-intercept $(0, 9)$ using function notation.

Answer: $f(x) = -2x + 9$.

Example 2. Write the line with slope 4 and y-intercept $(0, -\frac{2}{9})$ using function notation.

Answer: $f(x) = 4x - \frac{2}{9}$.

Example 3. Write the line with slope 3 through $(2, 1)$ using function notation.

Answer: $f(x) = 3x - 5$.

Example 4. Write the function through $(-1, 5)$ and $(3, -3)$.

Answer: $f(x) = -2x + 3$.

Example 5. Write the line with slope -9 through $(7, 8)$ using function notation.

Answer: $f(x) = -9x + 71$.

Example 6. Write the line with slope 0 through $(-1, -1)$ using function notation.

Answer: $f(x) = -1$.

Mod 1 O Nonlinear Functions

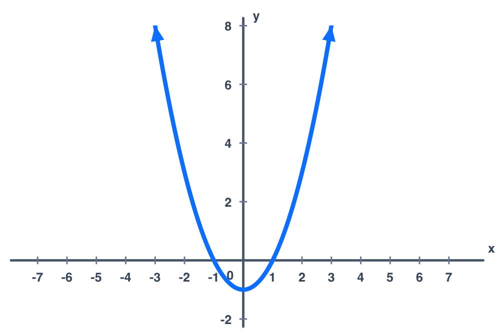
Graph basic quadratic, absolute-value, and square-root functions.

QUICK REFERENCE

- Quadratic, absolute-value, and square-root rules produce nonlinear graphs.
- A quadratic graph is a parabola; an absolute value graph is V-shaped.
- Square-root graphs have an endpoint and extend in one direction.
- Read horizontal and vertical shifts directly from the function before selecting or sketching its graph.
- Use several accurate points to confirm the graph's location and orientation.

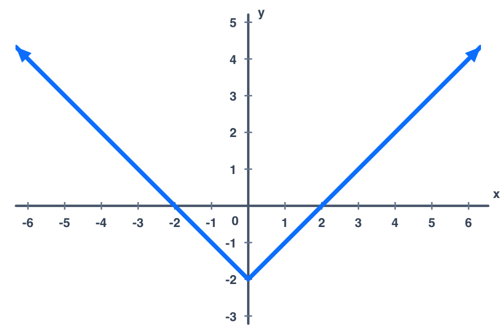
Example 1. Graph $f(x) = x^2 - 1$.

Answer: **The graph is an upward-opening parabola with vertex $(0, -1)$.**



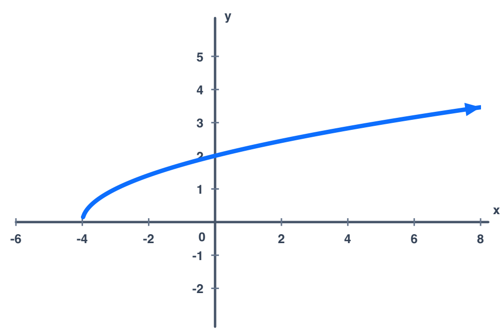
Example 2. Graph $g(x) = |x| - 2$.

Answer: **The graph is V-shaped with vertex** $(0, -2)$.



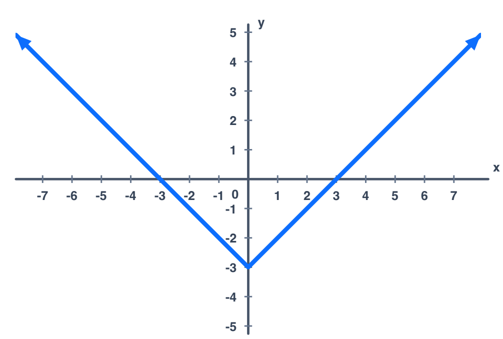
Example 3. Graph $h(x) = \sqrt{x+4}$.

Answer: **The graph begins at** $(-4, 0)$ **and extends right.**



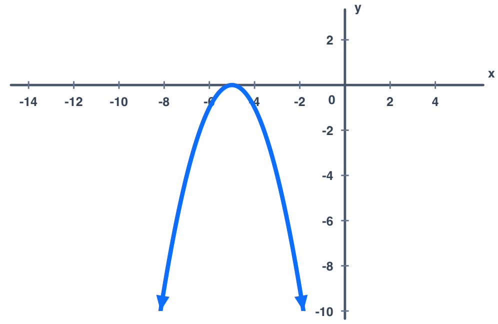
Example 4. Graph $p(x) = |x| - 3$.

Answer: **The graph is V-shaped with vertex** $(0, -3)$.



Example 5. Graph $q(x) = -(x+5)^2$.

Answer: **The graph is a downward-opening parabola with vertex $(-5, 0)$.**



Mod 1 P Piecewise Functions

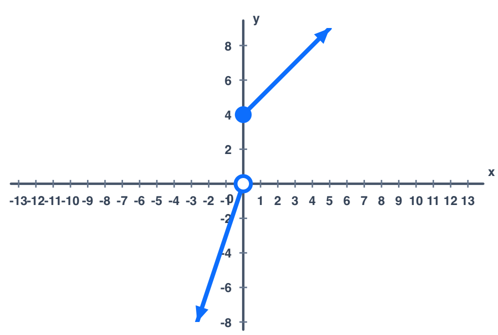
Graph piecewise functions and determine domain and range.

QUICK REFERENCE

- A piecewise function uses different rules on different parts of its domain.
- Graph each rule only over the interval named by its condition.
- Use a closed endpoint when the boundary value is included and an open endpoint when it is excluded.
- At a breakpoint, only one output can be included if the relation is a function.
- Read domain and range from every piece, including gaps and endpoints.

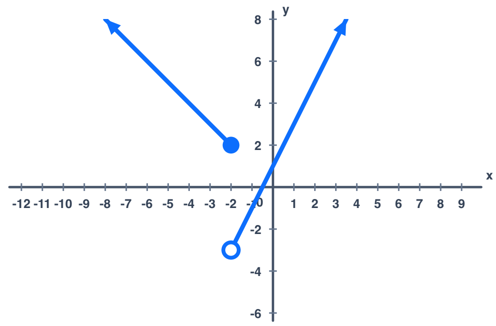
Example 1. Graph $f(x) = \begin{cases} 3x, & x < 0 \\ x+4, & x \geq 0 \end{cases}$.

Answer: **The first piece approaches an open point at $(0, 0)$; the second begins with a closed point at $(0, 4)$.**



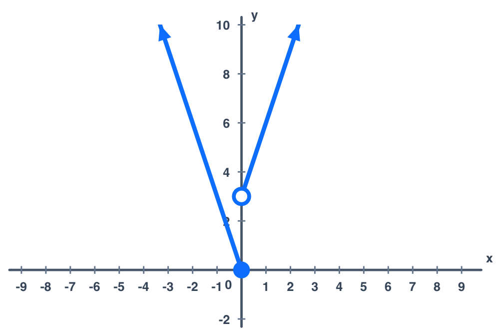
Example 2. Graph $g(x) = \begin{cases} -x, & x \leq -2 \\ 2x+1, & x > -2 \end{cases}$.

Answer: The first piece ends with a closed point at $(-2, 2)$; the second approaches an open point at $(-2, -3)$.



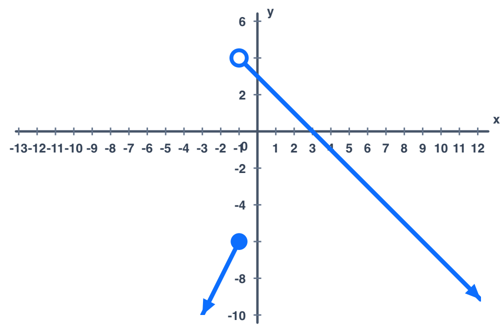
Example 3. Graph $h(x) = \begin{cases} -3x, & x \leq 0 \\ 3x+3, & x > 0 \end{cases}$, then state its domain and range.

Answer: Domain: $(-\infty, \infty)$. Range: $[0, \infty)$.



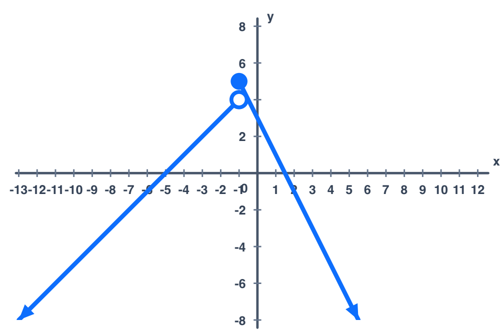
Example 4. Graph $p(x) = \begin{cases} 2x - 4, & x \le -1 \\ -x + 3, & x > -1 \end{cases}$, then state its domain and range.

Answer: **Domain:** $(-\infty, \infty)$. **Range:** $(-\infty, 4)$.



Example 5. Graph $q(x) = \begin{cases} x + 5, & x < -1 \\ -2x + 3, & x \ge -1 \end{cases}$, then state its domain and range.

Answer: **Domain:** $(-\infty, \infty)$. **Range:** $(-\infty, 5]$.



Module 2: Exponents, Polynomials, and Quadratic Foundations

Use exponent rules, operate on and divide polynomials, factor completely, and solve foundational quadratic equations and applications.

Mod 2 A Exponent Rules

Simplify expressions using the product, quotient, and power rules.

QUICK REFERENCE

- For the same base, multiply by adding exponents: $a^m a^n = a^{m+n}$.
- Divide same bases by subtracting exponents: $a^m/a^n = a^{m-n}$, for $a \neq 0$.
- A power raised to a power multiplies exponents: $(a^m)^n = a^{mn}$.
- Apply a power to every factor in a product or quotient.
- Simplify numerical coefficients and write the result with positive exponents.

Example 1. Simplify $(8y^3)(7y)$.

Answer: $56y^4$.

Example 2. Simplify $(-9xy^4)(4x^4y^5)$.

Answer: $-36x^5y^9$.

Example 3. Simplify $(4c^7)^3$.

Answer: $64c^{21}$.

Example 4. Simplify $\left(\dfrac{5x^5z^3}{y^2}\right)^2$.

Answer: $\dfrac{25x^{10}z^6}{y^4}$.

Example 5. Simplify $\dfrac{p^3q^7}{p^2q^2}$.

Answer: pq^5.

Example 6. Simplify $(-3xyz^4)^2$.

Answer: $9x^2y^2z^8$.

Example 7. Simplify $\left(\dfrac{4u^2}{8v^4}\right)^5$.

Answer: $\dfrac{u^{10}}{32v^{20}}$.

Mod 2 B Polynomial Introduction

Find polynomial degree and classify polynomials by their number of terms.

QUICK REFERENCE

- A polynomial is made of terms with numerical coefficients and nonnegative whole-number exponents.

- The degree of a one-variable polynomial is its greatest exponent.
- The degree is the greatest term degree; for multiple variables, add the exponents within a term.
- A monomial has one term, a binomial two, and a trinomial three.
- A polynomial with more than three terms is classified as none of these.

Example 1. Find the degree and classify $x^3 + 6$.

Answer: **Degree 3; binomial.**

Example 2. Find the degree and classify $6m^4 - 5m^3 + 7m - 8$.

Answer: **Degree 4; neither monomial, binomial, nor trinomial.**

Example 3. Find the degree of $4zx + 2z^3x^2 + 3zx^3$.

Answer: **Degree 5, from $2z^3x^2$.**

Example 4. Find the degree and classify $6 - 4x^3$.

Answer: **Degree 3; binomial.**

Example 5. Find the degree and classify $3xy^4 - 3$.

Answer: **Degree 5; binomial.**

Example 6. Find the degree and classify $2a^8 + 4a - 5$.

Answer: **Degree 8; trinomial.**

Mod 2 C Polynomial Operations

Add, subtract, and multiply polynomials, including common binomial products.

QUICK REFERENCE

- Add and subtract polynomials by combining like terms.
- When subtracting a polynomial, distribute the negative sign to every term.
- Multiply each term by every term in the other polynomial.
- For a binomial square, multiply the binomial by itself and combine like terms.
- Write the simplified result in standard form.

Example 1. Add $(-2x + 10) + (7x^2 + 2x + 12)$.

Answer: $7x^2 + 22$.

Example 2. Subtract $(7y^2 + 8y - 5) - (-2y + 2)$.

Answer: $7y^2 + 10y - 7$.

Example 3. Multiply $-2a^2(5a^2 - 2a + 5)$.

Answer: $-10a^4 + 4a^3 - 10a^2$.

Example 4. Multiply $(x+2)(x+7)$.

Answer: $x^2+9x+14$.

Example 5. Expand $(5y+5)^2$.

Answer: $25y^2+50y+25$.

Example 6. Multiply $(x-2)(x^2-7x+3)$.

Answer: $x^3-9x^2+17x-6$.

Example 7. Multiply $(x+4)(x^3-3x+5)$.

Answer: $x^4+4x^3-3x^2-7x+20$.

Mod 2 D Scientific Notation

Convert between standard and scientific notation for very large and very small numbers.

QUICK REFERENCE

- Scientific notation has the form $a \times 10^n$, where $1 \le |a| < 10$.
- A positive exponent moves the decimal right when returning to standard notation.
- A negative exponent moves the decimal left when returning to standard notation.
- Count decimal-place moves carefully when converting a standard number to scientific notation.
- Include placeholder zeros when writing a small number in standard notation.

Example 1. Write $43{,}000$ in scientific notation.

Answer: 4.3×10^4.

Example 2. Write 0.00000161 in scientific notation.

Answer: 1.61×10^{-6}.

Example 3. Write 1.667×10^{-7} in standard notation.

Answer: 0.0000001667.

Example 4. Write 6.1×10^{-2} in standard notation.

Answer: 0.061.

Example 5. Write 2.802×10^4 in standard notation.

Answer: $28{,}020$.

Example 6. A star converts 3.0×10^7 tons of hydrogen each second. Write that amount in standard notation.

Answer: $30{,}000{,}000$ tons.

Mod 2 E Polynomial Division

Divide polynomials by monomials and binomials using termwise division and long division.

QUICK REFERENCE

- Divide every numerator term when the divisor is a monomial.
- For long division, arrange both polynomials in descending powers and include zero placeholders for missing terms.
- Divide leading terms, multiply, subtract, and bring down the next term.
- Write a nonzero remainder over the divisor.
- Check with $\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$.

Example 1. Divide $\dfrac{9p^4 + 15p^3}{3p}$.

Answer: $3p^3 + 5p^2$.

Example 2. Divide $\dfrac{-4x^7 + 12x^5 - 8}{4x^2}$.

Answer: $-x^5 + 3x^3 - \dfrac{2}{x^2}$.

Example 3. Divide $\dfrac{x^2 + 9x + 20}{x + 5}$.

Answer: $x + 4$.

Example 4. Divide $\dfrac{6x^2 - 16x + 2}{x - 3}$.

Answer: $6x + 2 + \dfrac{8}{x - 3}$.

Example 5. Divide $\dfrac{2x^3 + 3x - 5}{x + 2}$.

Answer: $2x^2 - 4x + 11 - \dfrac{27}{x + 2}$.

Example 6. Check the division $x^3 - 1 = (x - 1)(x^2 + x + 1)$.

Answer: **Multiplying the divisor and quotient gives $x^3 - 1$, so the remainder is 0.**

Mod 2 F Synthetic Division

Use synthetic division for divisors of the form x minus c.

QUICK REFERENCE

- Synthetic division applies when the divisor is $x - c$. Use c in the synthetic setup.
- For $x + a$, the synthetic value is $-a$.
- List every coefficient in descending order, using zero for missing powers.
- Bring down, multiply, and add repeatedly.
- The final value is the remainder; the other entries are quotient coefficients.

Example 1. Use synthetic division to divide $x^2 + 3x - 28$ by $x + 7$.

Answer: $x - 4$.

Example 2. Use synthetic division to divide $x^3 - 9x^2 - 6x + 5$ by $x - 3$.

Answer: $x^2 - 6x - 24 - \dfrac{67}{x - 3}$.

Example 3. Use synthetic division to divide $17x^2 - 30$ by $x + 3$.

Answer: $17x - 51 + \dfrac{123}{x + 3}$.

Example 4. Use synthetic division to divide $4x^3 + 2x^2 - 4x + 1$ by $x - \dfrac{1}{2}$.

Answer: $4x^2 + 4x - 2$.

Example 5. Use synthetic division to divide $3x^3 + 15x^2 - 4x - 23$ by $x + 5$.

Answer: $3x^2 - 4 - \dfrac{3}{x + 5}$.

Mod 2 G The Remainder Theorem

Use the Remainder Theorem to evaluate polynomials at specified values.

QUICK REFERENCE

- When $P(x)$ is divided by $x - c$, the remainder is $P(c)$.
- Synthetic substitution computes $P(c)$ efficiently.
- Use every coefficient in descending order, including zero placeholders for missing powers.
- Bring down, multiply by c, and add to obtain the next entry.
- Direct substitution and synthetic division should give the same value.

Example 1. For $P(x) = x^3 + 5x^2 - 4x + 4$, find $P(4)$.

Answer: $P(4) = 132$.

Example 2. For $P(x) = 6x^3 - 6x^2 - 5x + 2$, find $P(-3)$.

Answer: $P(-3) = -199.$

Example 3. For $P(x) = 2x^4 - 4x^2 + 2$, find $P(-1)$.

Answer: $P(-1) = 0.$

Example 4. For $P(x) = x^5 - x^4 - x^3 - 5$, find $P\left(-\frac{1}{2}\right)$.

Answer: $P\left(-\frac{1}{2}\right) = -\frac{159}{32}.$

Example 5. For $P(x) = 3x^4 - 2x + 7$, find $P(2)$.

Answer: $P(2) = 51.$

Mod 2 H Factoring

Factor polynomials completely using GCF, grouping, trinomial patterns, and special products.

QUICK REFERENCE

- Always factor out the greatest common factor first.
- Four-term polynomials may factor by grouping.
- For $x^2 + bx + c$, find two numbers whose product is c and sum is b.
- A difference of squares factors as $a^2 - b^2 = (a-b)(a+b)$.
- If no integer factorization exists, classify the polynomial as prime.

Example 1. Factor by grouping: $x^3 + 7x^2 + 9x + 63$.

Answer: $(x+7)(x^2+9).$

Example 2. Factor completely: $2x^3 - 12x^2 + 16x$.

Answer: $2x(x-2)(x-4).$

Example 3. Factor by grouping: $15x^3 - 6x^2 + 10x - 4$.

Answer: $(5x-2)(3x^2+2).$

Example 4. Factor $x^2 - x - 72$.

Answer: $(x-9)(x+8).$

Example 5. Factor $6x^2 + x - 2$.

Answer: $(3x+2)(2x-1).$

Example 6. Factor $5m^2 + 11m + 15$, or state that it is prime.

Answer: The polynomial is prime.

Example 7. Factor $4x^2 - 81$.

Answer: $(2x - 9)(2x + 9)$.

Example 8. Factor $9x^2 + 4$, or state that it is prime.

Answer: The polynomial is prime.

Mod 2 I Solving Quadratic Equations by Factoring

Use the zero-product property to solve factorable quadratic equations.

QUICK REFERENCE

- Write the equation with zero on one side before factoring.
- Factor completely, then set each factor equal to zero.
- The zero-product property applies only when a product equals zero.
- A repeated factor gives a repeated solution that should be listed once.
- Check solutions in the original equation when expansion or rearrangement was required.

Example 1. Solve $(x - 1)(x + 1) = 0$.

Answer: $x = -1, 1$.

Example 2. Solve $x(x + 3) = 0$.

Answer: $x = -3, 0$.

Example 3. Solve $(5x - 8)(7x + 6) = 0$.

Answer: $x = \frac{8}{5}, -\frac{6}{7}$.

Example 4. Solve $x^2 - 12x + 32 = 0$.

Answer: $x = 4, 8$.

Example 5. Solve $x^2 - 2x = 24$.

Answer: $x = -4, 6$.

Example 6. Solve $10x^2 - 21x - 10 = 0$.

Answer: $x = -\frac{2}{5}, \frac{5}{2}$.

Example 7. Solve $4x^2 - 12x + 9 = 0$.

Answer: $x = \frac{3}{2}$**, a repeated solution.**

Mod 2 J Applications of Quadratic Equations

Translate applications into quadratic equations and choose solutions that fit the context.

QUICK REFERENCE

- Define the variable and write the quadratic model before solving.
- Projectile height in feet often uses $h(t) = -16t^2 + v_0 t + h_0$.
- Area applications use products of related dimensions.
- Set the model equal to the requested output, move all terms to one side, and factor when possible.
- Reject negative time, length, or quantity values that do not make sense.

Example 1. An object is thrown from a 96-foot building with initial velocity 80 ft/s. Its height is $h(t) = -16t^2 + 80t + 96$. When does it hit the ground?

Answer: $t = 6$ **seconds; reject** $t = -1$.

Example 2. A rectangle is 5 centimeters longer than its width and has area 84 square centimeters. Find its dimensions.

Answer: $w(w + 5) = 84$. **The width is** 7 **cm and the length is** 12 **cm.**

Example 3. The sides of a square are increased by 5 inches, producing an area of 169 square inches. Find the original side length.

Answer: $(x + 5)^2 = 169$. **The original side is** 8 **inches.**

Example 4. An object is dropped from 484 feet, so $h(t) = -16t^2 + 484$. When does it reach the ground?

Answer: $t = 5.5$ **seconds.**

Example 5. Manufacturing cost is $C(x) = x^2 + 15x + 55$. How many units give a cost of $\$10{,}505$?

Answer: **Solve** $x^2 + 15x + 55 = 10505$: $x = 95$ **units; reject** $x = -110$.

Module 3: Rational Expressions, Radicals, and Complex Numbers

Work with rational expressions and equations, build radical and rational-exponent skills, apply coordinate formulas, and operate with complex numbers.

Mod 3 A Domain of Rational Expressions

Find domain restrictions by identifying values that make a rational denominator zero.

QUICK REFERENCE

- A rational expression is undefined wherever its denominator equals zero.
- Set the denominator equal to zero and solve to find excluded values.
- Factor a polynomial denominator completely so every restriction is visible.
- Restrictions come from the original denominator, even if a factor later cancels.
- State the domain with exclusions, set-builder notation, or interval notation.

Example 1. Find the domain of $f(x) = \dfrac{5x-6}{7}$.

Answer: All real numbers, $(-\infty, \infty)$.

Example 2. Find the domain of $f(x) = \dfrac{5x}{5x+8}$.

Answer: $x \neq -\dfrac{8}{5}$.

Example 3. Find the domain of $R(x) = \dfrac{4+6x}{x^3+2x^2-3x}$.

Answer: The denominator is $x(x+3)(x-1)$, so $x \neq -3, 0, 1$.

Example 4. Find the domain of $C(x) = \dfrac{x-3}{x^2-16}$.

Answer: $x \neq -4, 4$.

Example 5. Find the domain of $g(t) = \dfrac{t+2}{t^2+9}$.

Answer: All real numbers because t^2+9 is never zero for real t.

Mod 3 B Simplifying Rational Expressions

Factor and simplify rational expressions while preserving restrictions from the original expression.

QUICK REFERENCE

- Factor numerators and denominators before canceling.
- Cancel common factors, not individual terms joined by addition or subtraction.
- Opposite binomials differ by a factor of -1: $a-b = -(b-a)$.
- Keep every excluded value from the original denominator.
- A rational expression is simplified when numerator and denominator share no nonconstant factor.

Example 1. Simplify $\dfrac{x+6}{6+x}$.

Answer: 1, with $x \neq -6$.

Example 2. Simplify $\dfrac{x-8}{8-x}$.

Answer: -1, with $x \neq 8$.

Example 3. Simplify $\dfrac{5}{15x+75}$.

Answer: $\dfrac{1}{3(x+5)}$, with $x \neq -5$.

Example 4. Simplify $\dfrac{-4x+4y}{x-y}$.

Answer: -4, with $x \neq y$.

Example 5. Simplify $\dfrac{3x^2-x-4}{3x-4}$.

Answer: $x+1$, with $x \neq \dfrac{4}{3}$.

Example 6. Simplify $\dfrac{x^3+7x^2}{x^2+3x-28}$.

Answer: $\dfrac{x^2}{x-4}$, with $x \neq -7, 4$.

Mod 3 C Multiplying and Dividing Rational Expressions

Multiply and divide rational expressions by factoring, canceling, and using reciprocals.

QUICK REFERENCE

- Factor every numerator and denominator before simplifying.
- To divide rational expressions, multiply by the reciprocal of the divisor.
- Cancel common factors across the complete product.
- Record restrictions from every original denominator and from values that make a divisor zero.
- Write final exponents as positive values.

Example 1. Multiply $\dfrac{5x^2}{y} \cdot \dfrac{2y}{9x}$.

Answer: $\dfrac{10x}{9}$, with $x \neq 0$ and $y \neq 0$.

Example 2. Multiply $\dfrac{30x^5}{x^9} \cdot \dfrac{6x^5}{5}$.

Answer: $36x$, with $x \neq 0$.

Example 3. Multiply $\dfrac{x}{7x-14} \cdot \dfrac{x^2-2x}{6}$.

Answer: $\dfrac{x^2}{42}$, with $x \neq 2$.

Example 4. Multiply $\dfrac{y^2+8y+15}{y^2+6y-40} \cdot \dfrac{y^2+2y-24}{y^2+6y+9}$.

Answer: $\dfrac{(y+5)(y+6)}{(y+10)(y+3)}$, with $y \neq -10, -3, 4$.

Example 5. Divide $\dfrac{7z^7}{3z^9} \div \dfrac{35z}{6z^4}$.

Answer: $\dfrac{2z}{5}$, with $z \neq 0$.

Example 6. Divide $\dfrac{x^2-9}{x^2-4x+3} \div \dfrac{x+3}{x-1}$.

Answer: 1, with $x \neq -3, 1, 3$.

Mod 3 D

Adding and Subtracting Rational Expressions With Like Denominators

Combine rational expressions that already have a common denominator.

QUICK REFERENCE

- When denominators match, add or subtract only the numerators.
- Use parentheses around an entire numerator when subtracting.
- Keep the common denominator, then factor and simplify if possible.
- Do not add denominators.
- Retain restrictions from the original denominator.

Example 1. Add $\dfrac{6m}{5n} + \dfrac{4m}{5n}$.

Answer: $\dfrac{2m}{n}$, with $n \neq 0$.

Example 2. Subtract $\dfrac{4x}{2x-7} - \dfrac{14}{2x-7}$.

Answer: 2, with $x \neq \dfrac{7}{2}$.

Example 3. Add $\dfrac{2}{y+4} + \dfrac{y+1}{y+4}$.

Answer: $\dfrac{y+3}{y+4}$, with $y \neq -4$.

Example 4. Subtract $\dfrac{7x^2+6x}{x-4} - \dfrac{23x+44}{x-4}$.

Answer: $7x+11$, with $x \neq 4$.

Example 5. Add $\dfrac{6z+1}{z-1} + \dfrac{5z+2}{z-1}$.

Answer: $\dfrac{11z+3}{z-1}$, with $z \neq 1$.

Mod 3 E

Adding and Subtracting Rational Expressions With Unlike Denominators

Find least common denominators and combine rational expressions with different denominators.

QUICK REFERENCE

- Factor every denominator before choosing the least common denominator.
- The LCD contains each distinct factor raised to its greatest needed power.
- Multiply each fraction by the missing factors, then combine numerators.
- Distribute subtraction signs before combining like terms.
- Factor the resulting numerator and simplify only after the fractions are combined.

Example 1. Subtract $\dfrac{8a}{b} - \dfrac{b}{4}$.

Answer: $\dfrac{32a-b^2}{4b}$, with $b \neq 0$.

Example 2. Add $\dfrac{6}{x} + \dfrac{7}{9x^2}$.

Answer: $\dfrac{54x+7}{9x^2}$, with $x \neq 0$.

Example 3. Subtract $\dfrac{3}{x+2} - \dfrac{2x}{x^2-4}$.

Answer: $\dfrac{x-6}{(x-2)(x+2)}$, with $x \neq -2, 2$.

Example 4. Add $\dfrac{3}{5x} + \dfrac{6}{x+7}$.

Answer: $\dfrac{33x+21}{5x(x+7)}$, with $x \neq 0, -7$.

Example 5. Subtract $\dfrac{x}{x^2-4} - \dfrac{5}{x^2-4x+4}$.

Answer: $\dfrac{x(x-2)-5(x+2)}{(x-2)^2(x+2)} = \dfrac{x^2-7x-10}{(x-2)^2(x+2)}$, with $x \neq -2, 2$.

Example 6. Add $\dfrac{x+8}{x^2+3x-28}+\dfrac{x+9}{x^2+5x-14}$.

Answer: $\dfrac{2x^2+11x-52}{(x+7)(x-4)(x-2)}$, **with** $x \neq -7, 2, 4$.

Mod 3 F Rational Equations

Solve equations containing rational expressions and check restrictions and extraneous candidates.

QUICK REFERENCE

- List excluded values before clearing denominators.
- Multiply every term by the least common denominator to remove fractions.
- Solve the resulting equation, which may be linear or quadratic.
- Reject any candidate excluded from the original equation.
- An equation may have one solution, multiple solutions, or no solution.

Example 1. Solve $\dfrac{x}{4}+\dfrac{3x}{12}=\dfrac{x}{24}+5$.

Answer: $x=12$.

Example 2. Solve $4-\dfrac{8}{x}=16$.

Answer: $x=-\dfrac{2}{3}$.

Example 3. Solve $6+\dfrac{x+9}{x}=10$.

Answer: $x=3$.

Example 4. Solve $\dfrac{15}{3y-8}=-1$.

Answer: $y=-\dfrac{7}{3}$.

Example 5. Solve $\dfrac{7}{y-10}=\dfrac{9y}{2y(y-10)}$.

Answer: **No solution. The algebra gives $y=0$, but 0 is excluded.**

Example 6. Solve $\dfrac{2y}{2y+6}+\dfrac{4y-18}{3y+9}=\dfrac{8y+11}{y+3}$.

Answer: **No solution. Clearing denominators gives $y=-3$, but -3 is excluded.**

Mod 3 G Solving Rational Equations With More Than One Variable

Rearrange rational formulas for a specified variable.

QUICK REFERENCE

- Treat all variables except the requested one as known constants.
- Clear fractional forms by multiplying through by common denominators.
- Collect every term containing the requested variable on one side.
- Factor out the requested variable when it appears in multiple terms.
- State restrictions needed for any denominator in the final formula.

Example 1. Solve $R = \dfrac{B}{L}$ for L.

Answer: $L = \dfrac{B}{R}$, with $R \neq 0$.

Example 2. Solve $T = \dfrac{3R}{S} + \dfrac{U}{S}$ for S.

Answer: $S = \dfrac{3R+U}{T}$, with $T \neq 0$.

Example 3. Solve $y = \dfrac{75C}{d^2}$ for C.

Answer: $C = \dfrac{yd^2}{75}$.

Example 4. Solve $D = \dfrac{Q+K}{F}$ for F.

Answer: $F = \dfrac{Q+K}{D}$, with $D \neq 0$.

Example 5. Solve $C = \dfrac{\pi t^2}{9}$ for t, assuming $t \geq 0$.

Answer: $t = 3\sqrt{\dfrac{C}{\pi}}$.

Example 6. Solve $\dfrac{1}{s} + \dfrac{1}{6} = \dfrac{1}{c}$ for s.

Answer: $s = \dfrac{6c}{6-c}$, with $c \neq 0, 6$.

Mod 3 H Rational Equation Word Problems

Model reciprocal, work, motion, and proportion applications with rational equations.

QUICK REFERENCE

- For work problems, a worker completing one job in t hours has rate $1/t$ job per hour.

- Combined work rates add when people or machines work together.
- Motion applications use $t = d/r$ when equal travel times are compared.
- Reciprocal relationships often translate directly into rational equations.
- Check units, restrictions, and whether the final value makes sense in context.

Example 1. Six times the reciprocal of a number equals three times the reciprocal of 6. Find the number.

Answer: $\frac{6}{x} = \frac{3}{6}$**, so** $x = 12$**.**

Example 2. One worker can pour a slab in 3 hours and another in 2 hours. How long do they take together?

Answer: $\frac{1}{t} = \frac{1}{3} + \frac{1}{2}$**, so** $t = \frac{6}{5}$ **hour, or** 1.2 **hours.**

Example 3. A pilot flies 468 miles with the wind in the same time as 360 miles against it. The still-air speed is 230 mph. Find the wind speed.

Answer: $\frac{468}{230 + w} = \frac{360}{230 - w}$**, so** $w = 30$ **mph.**

Example 4. A mixture uses 12 teaspoons of concentrate for 3 gallons of water. How much water is needed for 16 teaspoons?

Answer: $\frac{12}{3} = \frac{16}{g}$**, so** $g = 4$ **gallons.**

Example 5. A custodian completes a job in 8 hours. With a second worker, the job takes 4 hours. How long would the second worker take alone?

Answer: $\frac{1}{8} + \frac{1}{x} = \frac{1}{4}$**, so** $x = 8$ **hours.**

Example 6. Two pumps fill a tank together in 6 hours. One pump alone takes 10 hours. How long does the other take alone?

Answer: $\frac{1}{10} + \frac{1}{x} = \frac{1}{6}$**, so** $x = 15$ **hours.**

Mod 3 I Square Roots and Cube Roots

Evaluate and simplify square and cube roots, including roots with variables.

QUICK REFERENCE

- The principal square root $\sqrt{a}$ is nonnegative.
- A negative number has no real square root, but negative numbers do have real cube roots.
- Recognize perfect square, cube, and fourth powers before taking a root.
- Apply the root to both the numerical coefficient and variable powers.
- Use the stated nonnegative or positive variable assumptions when simplifying variable roots.

Example 1. Find $\sqrt{225}$.

Answer: 15.

Example 2. Simplify $\sqrt{49x^6}$, assuming $x \geq 0$.

Answer: $7x^3$.

Example 3. Find $\sqrt[3]{64}$.

Answer: 4.

Example 4. Find $\sqrt[3]{-216}$.

Answer: -6.

Example 5. Simplify $\sqrt[4]{81x^{16}}$, assuming $x \geq 0$.

Answer: $3x^4$.

Example 6. Simplify $\sqrt[3]{-27x^{12}y^6}$.

Answer: $-3x^4y^2$.

Example 7. Classify $\sqrt[4]{-81}$ in the real number system.

Answer: **It is not a real number.**

Mod 3 J Evaluating Radical Functions

Evaluate square-root and cube-root functions at allowed inputs.

QUICK REFERENCE

- Substitute the input into the entire radicand before evaluating the root.
- An even-index radical requires a nonnegative radicand for real-valued functions.
- Odd-index radicals accept every real radicand.
- Keep the radical index attached to the function when substituting.
- Simplify perfect powers and leave other real outputs in exact radical form.

Example 1. If $f(x) = \sqrt{3x+7}$, find $f(0)$.

Answer: $f(0) = \sqrt{7}$.

Example 2. If $g(x) = \sqrt[3]{x-13}$, find $g(5)$.

Answer: $g(5) = -2$.

Example 3. If $g(x) = \sqrt[3]{x-11}$, find $g(-16)$.

Answer: $g(-16) = -3$.

Example 4. If $f(x) = \sqrt{5x+2}$, find $f(3)$.

Answer: $f(3) = \sqrt{17}$.

Example 5. For $h(t) = \sqrt[3]{2t+9}$, find $h(-4)$.

Answer: $h(-4) = 1$.

Mod 3 K Graphing Radical Functions

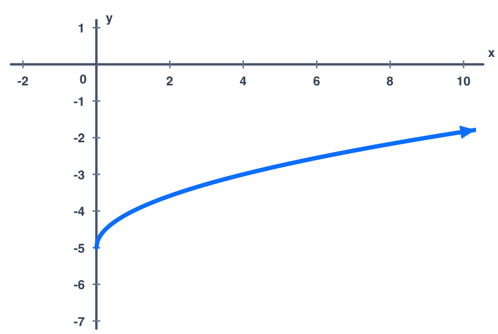
Find domains and graph square-root and cube-root functions from equations and tables.

QUICK REFERENCE

- For a square-root function, require the radicand to be nonnegative to find the domain.
- The parent square-root function begins at $(0, 0)$ and extends right.
- In $\sqrt{x-h} + k$, the endpoint is (h, k).
- Cube-root functions extend in both directions and have domain all real numbers.
- Use convenient inputs that make the radicand a perfect square or perfect cube when completing a table.

Example 1. Find the domain and graph $f(x) = \sqrt{x} - 5$.

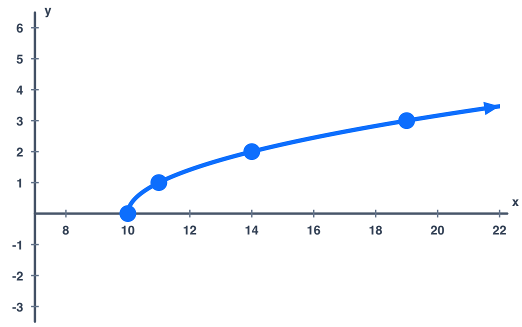
Answer: **Domain:** $[0, \infty)$. **Endpoint:** $(0, -5)$.



Example 2. Complete a table for $f(x) = \sqrt{x - 10}$ at $x = 10, 11, 14, 19$, then graph it.

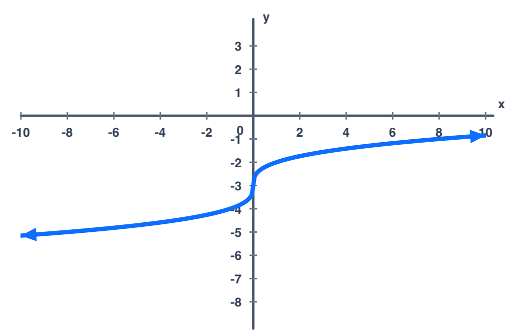
Answer: Outputs: $0, 1, 2, 3$. **Domain:** $[10, \infty)$.

x	10	11	14	19
$f(x)$	0	1	2	3



Example 3. Find the domain and graph $g(x) = \sqrt[3]{x} - 3$.

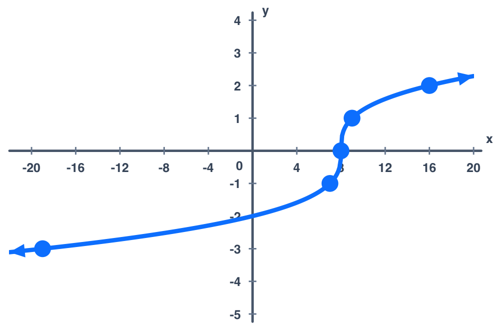
Answer: Domain: $(-\infty, \infty)$. **The graph passes through** $(0, -3)$.



Example 4. Complete a table for $h(x) = \sqrt[3]{x-8}$ at $x = 8, 16, 7, 9, -19$, then graph it.

Answer: **Outputs:** $0, 2, -1, 1, -3$. **Domain:** $(-\infty, \infty)$.

x	8	16	7	9	-19
$h(x)$	0	2	-1	1	-3



Mod 3 L Rational Exponents

Write rational exponents in radical notation and simplify products and quotients with rational exponents.

QUICK REFERENCE

- $a^{1/n} = \sqrt[n]{a}$ and $a^{m/n} = \sqrt[n]{a^m}$.
- The denominator of a rational exponent is the root index; the numerator is the power.
- Apply the product rule by adding rational exponents with a common denominator.
- Apply the quotient rule by subtracting rational exponents.
- Write the simplified result with a positive exponent.

Example 1. Write $(3x)^{2/3}$ in radical notation.

Answer: $\sqrt[3]{(3x)^2}$.

Example 2. Write $(5x+4)^{3/4}$ in radical notation.

Answer: $\sqrt[4]{(5x+4)^3}$.

Example 3. Simplify $c^{3/5}c^{8/5}$.

Answer: $c^{11/5}$.

Example 4. Simplify $x^{-6/5}x^{11/5}$, assuming $x > 0$.

Answer: x.

Example 5. Simplify $\dfrac{y^{1/5}}{y^{1/10}}$, assuming $y > 0$.

Answer: $y^{1/10}$.

Example 6. Simplify $\dfrac{a^{7/6}}{a^{1/3}}$, assuming $a > 0$.

Answer: $a^{5/6}$.

Mod 3 M Product and Quotient Rules for Radicals

Multiply, divide, and simplify radical expressions using product and quotient properties.

QUICK REFERENCE

- For appropriate real values, $\sqrt{a}\sqrt{b} = \sqrt{ab}$.
- The quotient rule gives $\sqrt{a/b} = \sqrt{a}/\sqrt{b}$ when defined.
- Multiply first, then extract perfect-power factors.
- The radical indices must match to use product or quotient rules directly.
- State variable assumptions or use absolute value when needed.

Example 1. Multiply $\sqrt{3} \cdot \sqrt{2}$.

Answer: $\sqrt{6}$.

Example 2. Multiply $\sqrt{125} \cdot \sqrt{45}$.

Answer: 75.

Example 3. Multiply $\sqrt[3]{3} \cdot \sqrt[3]{4}$.

Answer: $\sqrt[3]{12}$.

Example 4. Multiply $\sqrt{7x} \cdot \sqrt{3y}$, assuming positive variables.

Answer: $\sqrt{21xy}$.

Example 5. Simplify $\sqrt{\dfrac{2}{81}}$.

Answer: $\dfrac{\sqrt{2}}{9}$.

Example 6. Simplify $\sqrt{\dfrac{72x^5}{2x}}$, assuming $x \geq 0$.

Answer: $6x^2$.

Mod 3 N Distance Formula

Find exact and approximate distances between points in the coordinate plane.

QUICK REFERENCE

- The distance formula is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- Subtract coordinates in the same order; squaring removes sign differences.
- Simplify the radical for an exact answer before finding a decimal approximation.
- Horizontal distance is the absolute difference of x-coordinates; vertical distance uses y-coordinates.
- Distance is always nonnegative and should include units when given.

Example 1. Find the exact distance between $(-5, 3)$ and $(2, -3)$, then approximate to the nearest thousandth.

Answer: $\sqrt{85} \approx 9.220$ **units.**

Example 2. Find the distance between $(-12, 9)$ and $(-7, 2)$.

Answer: $\sqrt{74} \approx 8.602$ **units.**

Example 3. Find the distance between $(0, \frac{3}{2})$ and $(-3, \frac{2}{5})$.

Answer: $\frac{\sqrt{1021}}{10} \approx 3.195$ **units.**

Example 4. Find the distance between $(4, -2)$ and $(4, 11)$.

Answer: 13 **units.**

Example 5. A map uses coordinates $(2, 5)$ and $(10, 11)$ for two locations, with one coordinate unit equal to 3 miles. Find the actual distance.

Answer: Coordinate distance: 10 **units. Actual distance:** 30 **miles.**

Mod 3 O Midpoint Formula

Find the point halfway between two endpoints in the coordinate plane.

QUICK REFERENCE

- The midpoint formula is $M = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.
- Average the x-coordinates and y-coordinates separately.
- Keep fractional coordinates in simplified exact form.
- Average decimal coordinates in the same way as integer coordinates.
- The midpoint lies on the segment and is the same distance from both endpoints.

Example 1. Find the midpoint of $(4, -9)$ and $(6, 3)$.

Answer: $(5, -3)$.

Example 2. Find the midpoint of $(-2,-2)$ and $(-10,7)$.

Answer: $(-6,\frac{5}{2})$.

Example 3. Find the midpoint of $(7,3)$ and $(-2,-3)$.

Answer: $(\frac{5}{2},0)$.

Example 4. Find the midpoint of $(-\frac{5}{6},\frac{6}{7})$ and $(-\frac{7}{6},\frac{2}{7})$.

Answer: $(-1,\frac{4}{7})$.

Example 5. Find the midpoint of $(-5.5,4.2)$ and $(-9.2,0.7)$.

Answer: $(-7.35,2.45)$.

Mod 3 P Adding and Subtracting Radical Expressions

Simplify radicals and combine like radical terms.

QUICK REFERENCE

- Like radicals have the same index and the same simplified radicand.
- Simplify every radical before deciding which terms are like.
- Add or subtract coefficients while keeping the common radical part.
- Radicals with different simplified radicands cannot be combined.
- Apply the same process to variable radicals under the stated assumptions.

Example 1. Simplify $\sqrt{180}-\sqrt{20}$.

Answer: $4\sqrt{5}$.

Example 2. Simplify $9\sqrt{3x^3}+3x\sqrt{27x}$, assuming $x>0$.

Answer: $18x\sqrt{3x}$.

Example 3. Simplify $5\sqrt{75}-4\sqrt{50}+\sqrt{12}$.

Answer: $27\sqrt{3}-20\sqrt{2}$.

Example 4. Simplify $\sqrt{36b^3}+\sqrt{16b^3}-\sqrt{25b^3}$, assuming $b>0$.

Answer: $5b\sqrt{b}$.

Example 5. Simplify $-4\sqrt{625}+3\sqrt{40}$.

Answer: $-100+6\sqrt{10}$.

Example 6. A trapezoid has side lengths $7\sqrt{2}$, $2\sqrt{18}$, $\sqrt{18}$, and $2\sqrt{50}$ inches. Find its perimeter.

Answer: $7\sqrt{2}+6\sqrt{2}+3\sqrt{2}+10\sqrt{2}=26\sqrt{2}$ **inches.**

Mod 3 Q Multiplying Radical Expressions

Multiply monomial and binomial radical expressions and simplify the results.

QUICK REFERENCE

- Use distribution or FOIL when multiplying sums containing radicals.
- Multiply coefficients and radicands, then simplify.
- $(\sqrt{a})^2 = a$ when the principal root is defined.
- Conjugates $(a+b)(a-b)$ create a difference of squares.
- Combine like terms after all products are simplified.

Example 1. Multiply $7(\sqrt{10}+\sqrt{3})$.

Answer: $7\sqrt{10}+7\sqrt{3}$.

Example 2. Multiply $(-7\sqrt{2})(2\sqrt{2})$.

Answer: -28.

Example 3. Multiply $(2-\sqrt{3x})(9+2\sqrt{3x})$, assuming $x \geq 0$.

Answer: $18-5\sqrt{3x}-6x$.

Example 4. Multiply $(9-\sqrt{55})(9+\sqrt{55})$.

Answer: 26.

Example 5. Multiply $(\sqrt{5}+3)^2$.

Answer: $14+6\sqrt{5}$.

Example 6. Multiply $(2\sqrt{x}-\sqrt{y})(3\sqrt{x}+4\sqrt{y})$.

Answer: $6x+5\sqrt{xy}-4y$.

Mod 3 R Rationalizing Denominators

Rewrite radical fractions so no radical remains in the denominator.

QUICK REFERENCE

- For a binomial denominator containing a radical, multiply by its conjugate.
- Use the opposite middle sign when writing the conjugate.
- Conjugates have the same terms with the middle sign changed.
- The denominator becomes a difference of squares.

- Simplify the numerator and denominator completely after rationalizing.

Example 1. Rationalize $\dfrac{9}{2-\sqrt{7}}$.

Answer: $-6-3\sqrt{7}$.

Example 2. Rationalize $\dfrac{-8}{\sqrt{x}-3}$.

Answer: $\dfrac{-8(\sqrt{x}+3)}{x-9}$, with $x \geq 0$ and $x \neq 9$.

Example 3. Rationalize $\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$.

Answer: $7-4\sqrt{3}$.

Example 4. Rationalize $\dfrac{5}{1+\sqrt{3}}$.

Answer: $\dfrac{5(\sqrt{3}-1)}{2}$.

Example 5. Rationalize $\dfrac{6}{\sqrt{5}+2}$.

Answer: $6\sqrt{5}-12$.

Mod 3 S Solving Radical Equations

Isolate radicals, remove them with powers, and reject extraneous solutions.

QUICK REFERENCE

- Isolate one radical before raising both sides to a power.
- Square both sides to remove a square root; cube both sides to remove a cube root.
- Simplify and isolate again when an equation contains two radicals.
- Even powers can introduce extraneous solutions.
- Check every candidate in the original radical equation.

Example 1. Solve $\sqrt{2x}=4$.

Answer: $x=8$.

Example 2. Solve $\sqrt{x-3}=6$.

Answer: $x=39$.

Example 3. Solve $\sqrt{5x-9}-4=0$.

Answer: $x=5$.

Example 4. Solve $\sqrt{x+5} = x - 1$.

Answer: $x = 4$; the other algebraic candidate is extraneous.

Example 5. Solve $\sqrt{x+1} + 1 = \sqrt{2x+4}$.

Answer: $x = 0$.

Example 6. Solve $\sqrt[3]{2x-1} = 3$.

Answer: $x = 14$.

Example 7. Solve $\sqrt{x-2} = -3$.

Answer: No solution, $\varnothing$.

Mod 3 T Complex Numbers

Write and operate with complex numbers using the imaginary unit i.

QUICK REFERENCE

- The imaginary unit is defined by $i^2 = -1$, so $\sqrt{-a} = i\sqrt{a}$ for $a > 0$.
- Combine real parts with real parts and imaginary parts with imaginary parts.
- Multiply using distribution and replace i^2 with -1.
- To divide complex numbers, multiply by the conjugate of the denominator.
- Powers of i repeat every four exponents.

Example 1. Express $\sqrt{-121}$ in terms of i.

Answer: $11i$.

Example 2. Simplify $\sqrt{-63}$.

Answer: $3i\sqrt{7}$.

Example 3. Add $(7-8i) + (3+4i)$.

Answer: $10 - 4i$.

Example 4. Subtract $(2+6i) - (3-4i)$.

Answer: $-1 + 10i$.

Example 5. Multiply $(-7i)(-10i)$.

Answer: -70.

Example 6. Multiply $2i(5-2i)$.

Answer: $4 + 10i$.

Example 7. Write $\dfrac{4}{2+8i}$ in $a+bi$ form.

Answer: $\dfrac{2}{17}-\dfrac{8}{17}i.$

Example 8. Find $(-2i)^9$.

Answer: $-512i.$

Module 4: Quadratic Methods, Inequalities, Graphs, and Applications

Finish quadratic equation methods, solve polynomial and rational inequalities, graph quadratics in multiple forms, and interpret quadratic models.

Mod 4 A Solving Quadratic Equations Using the Square Root Method

Use the square-root property to solve equations with an isolated squared expression.

QUICK REFERENCE

- If $u^2 = k$ with $k > 0$, then $u = \pm\sqrt{k}$.
- Isolate the squared expression before taking square roots.
- Remember the $\pm$ when solving an equation; the principal radical symbol alone is nonnegative.
- Simplify exact radical answers before listing both real solutions.
- Check both candidates in the original equation.

Example 1. Solve $x^2 = 9$.

Answer: $x = -3, 3.$

Example 2. Solve $x^2 - 11 = 0$.

Answer: $x = \pm\sqrt{11}.$

Example 3. Solve $x^2 = 98$.

Answer: $x = \pm 7\sqrt{2}.$

Example 4. Solve $(x-4)^2 = 25$.

Answer: $x = -1, 9.$

Example 5. Solve $3(x+2)^2 = 42$.

Answer: $x = -2 \pm \sqrt{14}.$

Example 6. Solve $(3x+1)^2 = 8$.

Answer: $x = \dfrac{-1 \pm 2\sqrt{2}}{3}$.

Mod 4 B Solving Quadratic Equations by Completing the Square

Create perfect-square trinomials and solve quadratic equations by completing the square.

QUICK REFERENCE

- For $x^2 + bx$, add $(b/2)^2$ to create $(x + b/2)^2$.
- Add the same value to both sides of the equation.
- Rewrite the left side as a perfect-square binomial after adding $(b/2)^2$.
- Use the square-root property after rewriting one side as a perfect square.
- Keep real radical answers in exact simplified form.

Example 1. Solve $x^2 + 12x = -27$ by completing the square.

Answer: $x = -3, -9$.

Example 2. Solve $x^2 + 8x = -7$ by completing the square.

Answer: $x = -1, -7$.

Example 3. Solve $x^2 + 8x + 11 = 0$ by completing the square.

Answer: $x = -4 \pm \sqrt{5}$.

Example 4. Solve $x^2 - 6x + 1 = 0$ by completing the square.

Answer: $x = 3 \pm 2\sqrt{2}$.

Example 5. Solve $x^2 + 2x - 5 = 0$ by completing the square.

Answer: $x = -1 \pm \sqrt{6}$.

Example 6. Solve $x^2 + 6x + 6 = 0$ by completing the square.

Answer: $x = -3 \pm \sqrt{3}$.

Mod 4 C Solving Quadratic Equations Using the Quadratic Formula

Use the quadratic formula to solve quadratic equations.

QUICK REFERENCE

- For $ax^2 + bx + c = 0$, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- Write the equation in standard form and identify a, b, and c with their signs.

- Evaluate $b^2 - 4ac$ carefully before simplifying the square root.
- Simplify the radical and reduce the full fraction when possible.
- Use exact answers unless a decimal approximation is requested.

Example 1. Solve $m^2 - 4m - 12 = 0$ using the quadratic formula.

Answer: $m = -2, 6$.

Example 2. Solve $5y = 3y^2 - 8$.

Answer: $y = -1, \frac{8}{3}$.

Example 3. Solve $x^2 - 8x + 16 = 0$.

Answer: $x = 4$, a repeated solution.

Example 4. Solve $2x^2 + 3x - 7 = 0$.

Answer: $x = \frac{-3 \pm \sqrt{65}}{4}$.

Example 5. Solve $x^2 + 2x + 10 = 0$.

Answer: $x = -1 \pm 3i$.

Example 6. Solve $9m^2 - 2m = 10$ using the quadratic formula.

Answer: $m = \frac{1 \pm \sqrt{91}}{9}$.

Mod 4 D Solving Equations in Quadratic Form

Solve radical, rational, and quartic equations that lead to quadratic equations.

QUICK REFERENCE

- For a radical equation, isolate the radical and square both sides before solving the resulting quadratic.
- Check every radical-equation candidate because squaring can introduce an extraneous solution.
- For a rational equation, list excluded values and multiply through by the least common denominator.
- For a quartic in x^2, let $u = x^2$, solve the quadratic in u, and then solve for x.
- Check restrictions and every candidate in the original equation.

Example 1. Solve $3x = \sqrt{1 + 8x}$.

Answer: Squaring gives $9x^2 - 8x - 1 = 0$. The candidates are 1 and $-\frac{1}{9}$, but only $x = 1$ checks.

Example 2. Solve $\sqrt{100x} = x + 16$.

Answer: Squaring gives $x^2 - 68x + 256 = 0$, so $x = 4, 64$. Both solutions check.

Example 3. Solve $\dfrac{2}{x} + \dfrac{3}{x-3} = 1$.

Answer: The restrictions are $x \neq 0, 3$. Clearing denominators gives $x^2 - 8x + 6 = 0$, so $x = 4 \pm \sqrt{10}$.

Example 4. Solve $\dfrac{4}{x^2 - 3x + 2} = \dfrac{3x}{x-1} - \dfrac{x}{x-2}$.

Answer: The restrictions are $x \neq 1, 2$. Clearing denominators gives solutions $x = \dfrac{5 \pm \sqrt{57}}{4}$.

Example 5. Solve $p^4 - 16 = 0$.

Answer: $p = -2, 2, -2i, 2i$.

Example 6. Solve $z^4 - 20z^2 + 64 = 0$.

Answer: Let $u = z^2$. Then $(u-4)(u-16) = 0$, so $z = -4, -2, 2, 4$.

Mod 4 E Polynomial and Rational Inequalities

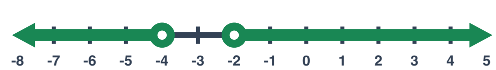
Use critical values and sign analysis to solve polynomial and rational inequalities.

QUICK REFERENCE

- Move all terms to one side and factor when possible.
- Critical values are zeros of the numerator and, for rational expressions, excluded zeros of the denominator.
- Use critical values to divide the number line into test intervals.
- Include zeros for $\leq$ or $\geq$, but never include denominator zeros.
- Express the selected sign intervals with interval notation.

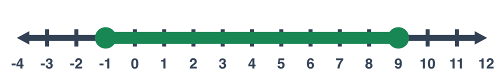
Example 1. Solve $(x+4)(x+2) > 0$.

Answer: $(-\infty, -4) \cup (-2, \infty)$.



Example 2. Solve $(x-9)(x+1) \leq 0$.

Answer: $[-1, 9]$.



Example 3. Solve $x^2 + 2x - 3 \le 0$.

Answer: $[-3, 1]$.

Example 4. Solve $x(x-2)(x+4) \le 0$.

Answer: $(-\infty, -4] \cup [0, 2]$.

Example 5. Solve $\dfrac{x-3}{x+2} > 0$.

Answer: $(-\infty, -2) \cup (3, \infty)$.

Example 6. Solve $\dfrac{(x+1)(x-4)}{x-2} \le 0$.

Answer: $(-\infty, -1] \cup (2, 4]$.

Example 7. Solve $\dfrac{x+5}{(x-1)^2} \ge 0$.

Answer: $[-5, 1) \cup (1, \infty)$.

Mod 4 F Graphing Quadratic Functions in Vertex Form

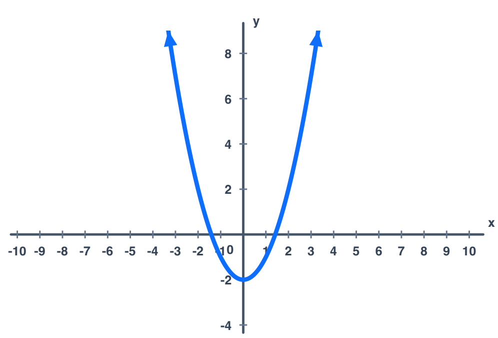
Graph quadratics in vertex form and identify the vertex and axis of symmetry.

QUICK REFERENCE

- Vertex form is $f(x) = a(x-h)^2 + k$, with vertex (h, k).
- The axis of symmetry is $x = h$.
- If $a > 0$, the parabola opens up; if $a < 0$, it opens down.
- Plot the vertex first and use symmetric points on both sides of the axis.
- Draw the parabola as a solid curve and the axis of symmetry as a dashed vertical line.

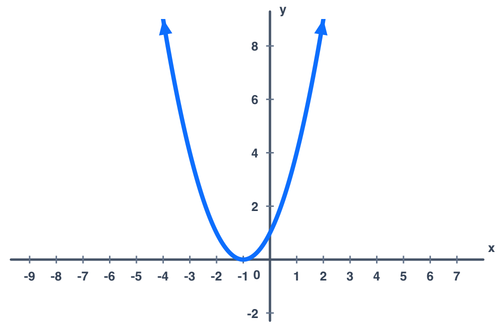
Example 1. Graph $f(x) = x^2 - 2$. State the vertex and axis of symmetry.

Answer: **Vertex:** $(0, -2)$. **Axis:** $x = 0$. **Opens upward.**



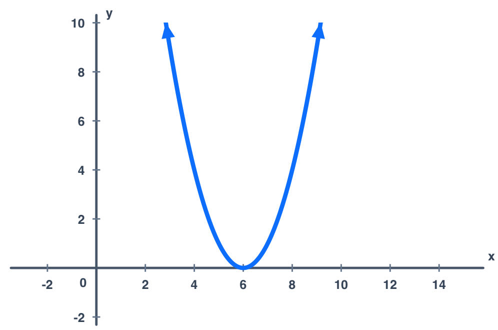
Example 2. Graph $h(x) = (x+1)^2$. State the vertex and axis.

Answer: Vertex: $(-1, 0)$. **Axis:** $x = -1$.



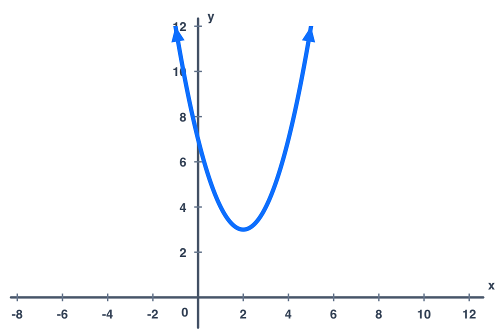
Example 3. Graph $g(x) = (x-6)^2$. State the vertex and axis of symmetry.

Answer: Vertex: $(6, 0)$. **Axis:** $x = 6$.



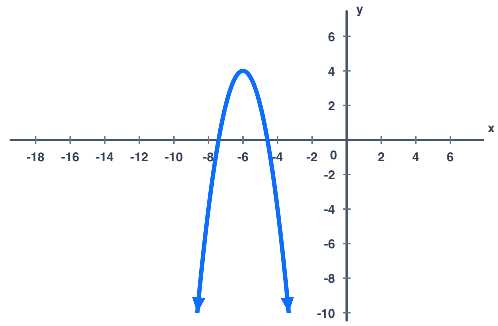
Example 4. Graph $p(x) = (x-2)^2 + 3$. State the vertex and axis of symmetry.

Answer: Vertex: $(2, 3)$. **Axis:** $x = 2$.



Example 5. Graph $q(x) = -2(x+6)^2 + 4$. State the vertex and axis of symmetry.

Answer: Vertex: $(-6, 4)$. **Axis:** $x = -6$.



Mod 4 G Graphing Quadratic Functions Continued

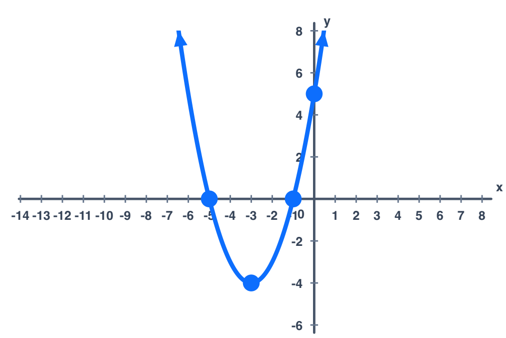
Analyze and graph quadratic functions in standard form using vertices, intercepts, and symmetry.

QUICK REFERENCE

- For $f(x) = ax^2 + bx + c$, the axis of symmetry is $x = -b/(2a)$.
- Substitute the axis value to find the vertex.
- The y-intercept is $(0, c)$; solve $ax^2 + bx + c = 0$ for x-intercepts.
- Use symmetry to plot matching points on opposite sides of the axis.
- Use the sign of a to determine whether the graph opens upward or downward.

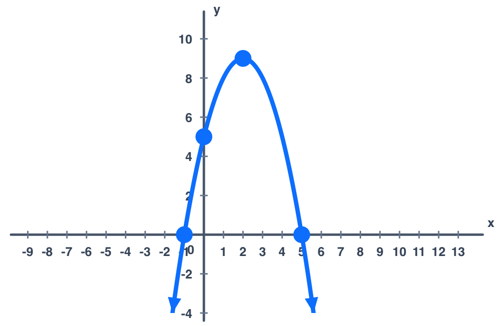
Example 1. Analyze and graph $f(x) = x^2 + 6x + 5$.

Answer: Vertex: $(-3, -4)$. **Axis:** $x = -3$. **Opens up. x-intercepts:** $(-5, 0)$, $(-1, 0)$. **y-intercept:** $(0, 5)$.



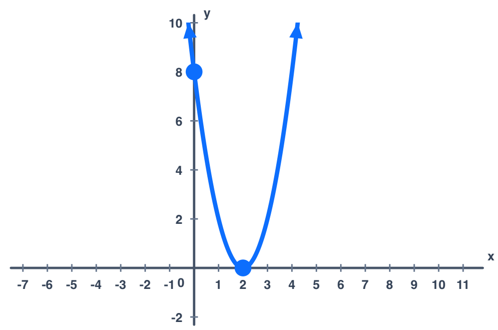
Example 2. Analyze and graph $g(x) = -x^2 + 4x + 5$.

Answer: Vertex: $(2, 9)$. **Axis:** $x = 2$. **Opens down. x-intercepts:** $(-1, 0)$, $(5, 0)$. **y-intercept:** $(0, 5)$.



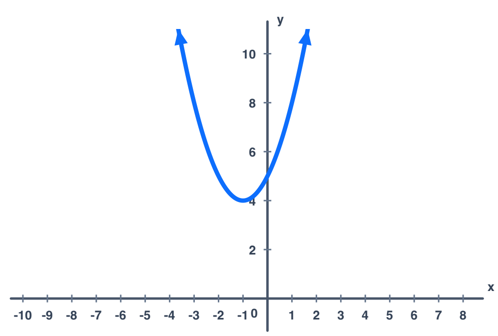
Example 3. Analyze and graph $h(x) = 2x^2 - 8x + 8$.

Answer: Vertex: $(2, 0)$. **Axis:** $x = 2$. **Opens up. Repeated x-intercept:** $(2, 0)$. **y-intercept:** $(0, 8)$.



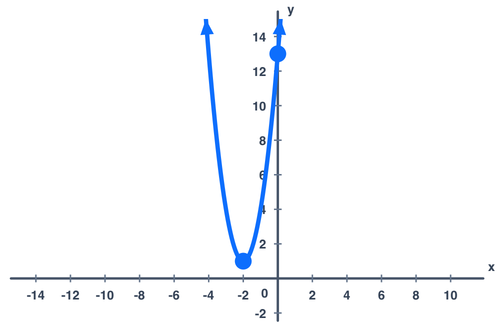
Example 4. Analyze $q(x) = x^2 + 2x + 5$. Does it have real x-intercepts?

Answer: Vertex: $(-1, 4)$. **It opens up and has no real x-intercepts because its minimum value is** 4.



Example 5. Analyze and graph $r(x) = 3x^2 + 12x + 13$.

Answer: **Vertex:** $(-2, 1)$**. Axis:** $x = -2$**. Opens up. There are no real x-intercepts. y-intercept:** $(0, 13)$**.**



Mod 4 H

Applications of Quadratic Functions

Use quadratic vertices to solve maximum and minimum application problems.

QUICK REFERENCE

- A quadratic vertex gives a maximum when the parabola opens down and a minimum when it opens up.
- Use $x = -b/(2a)$ to find the input at the vertex, then evaluate the model.
- Projectile models use the vertex height to answer maximum-height questions.
- Cost, number-product, and rectangle-area models require units and a contextual interpretation.
- Restrict the domain to values that make sense and reject nonphysical answers.

Example 1. A projectile is launched from the ground at 96 ft/s, with height $h(t) = -16t^2 + 96t$. Find its maximum height.

Answer: **The maximum height is** 144 **feet.**

Example 2. Manufacturing cost is $C(x) = 4x^2 - 1600x + 170{,}500$. How many units minimize cost, and what is the minimum cost?

Answer: **The minimum occurs at** $x = 200$ **units, with cost** $\$10{,}500$.

Example 3. Find two numbers whose sum is 82 and whose product is as large as possible.

Answer: **The numbers are** 41 **and** 41**; their maximum product is** 1681.

Example 4. Find two numbers whose difference is 56 and whose product is as small as possible.

Answer: **The numbers are** 28 **and** -28**; their minimum product is** -784.

Example 5. The length and width of a rectangle have a sum of 44. Find the dimensions that maximize its area.

Answer: **The rectangle is** 22 **by** 22**, with maximum area** 484 **square units.**

Example 6. Find two numbers whose sum is 60 and whose product is as large as possible.

Answer: The numbers are 30 and 30; their maximum product is 900.