College Algebra

A function-focused algebra companion with concise notes, worked examples, and quick references for class, study, and review.

Module 1: Function Introduction

Students build function language, interval notation, graph-feature vocabulary, parent-function recognition, and transformation language from tables, formulas, and graphs.

HW 1 Function Basics and Notation

Read functions from tables, formulas, and graphs while tracking notation, domain, range, zeros, and intercepts.

QUICK REFERENCE

- A function gives each input exactly one output. A relation is not a function if one input is paired with two different outputs.
- The vertical line test checks a graph: if one vertical line hits the graph more than once, the graph is not a function.
- Function notation such as $f(-2)$ means "the output when the input is -2." It does not mean multiplication.
- For a function, y and $f(x)$ can both name the output. For example, $y = 2x - 3$ and $f(x) = 2x - 3$ can describe the same rule.
- A function can be shown by a formula, table, graph, mapping, or words.
- Domain is the set of allowed inputs. Range is the set of outputs the function actually gives.
- Use interval notation to describe domain and range.
  - Use brackets when an endpoint is included and parentheses when it is not included.
  - Use $-\infty$ and ∞ only with parentheses.
  - Use $\cup$ when separate intervals need to be joined.
- Zeros are input values where $f(x) = 0$. On a graph, real zeros are x-intercepts. The y-intercept happens when $x = 0$.

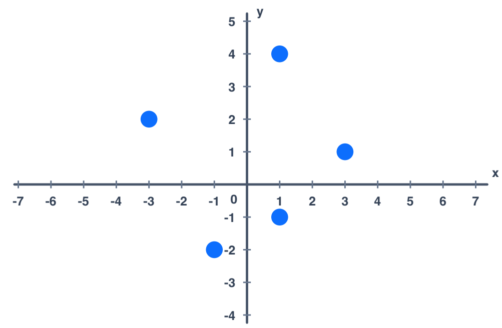
Example 1. Decide whether each relation is a function: A: $(-2, 4), (0, -1), (3, 0), (5, 7)$; B: $(1, 3), (1, 5), (2, 6), (4, 8)$.

Answer: A is a function. B is not a function because input 1 has two outputs.

Example 2. If $h(t) = 4t - 7$, find $h(-2)$, $h(0)$, $h(5)$, and $h(7/4)$. What is unique about $h(0)$ and $h(7/4)$?

Answer: $h(-2) = -15$, $h(0) = -7$, $h(5) = 13$, and $h(7/4) = 0$. The value $h(0)$ represents the y-intercept of the function and $h(7/4)$ represents a zero (x-intercept) of the function.

Example 3. Use the plotted relation to decide whether it is a function. Explain using the vertical line test.



Answer: It is not a function. The input $x = 1$ is paired with two outputs, so a vertical line at $x = 1$ hits the relation twice.

Example 4. Evaluate the non-linear functions $f(x) = x^2 - 2x$ and $g(x) = \sqrt{x+4}$. Find $f(-1)$, $f(3)$, and $g(5)$.

Answer: $f(-1) = 3$, $f(3) = 3$, and $g(5) = 3$.

Example 5. Complete the table for the linear function $L(x) = 2x - 3$ and the quadratic function $Q(x) = x^2 - 4$. Identify the y-intercepts that appear in the table.

x	-2	0	3
$L(x) = 2x - 3$			
$Q(x) = x^2 - 4$			

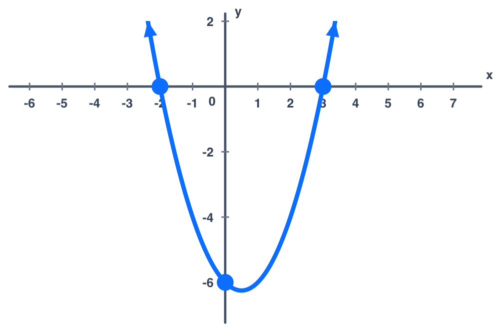
Answer: For L, the outputs are $-7, -3, 3$. For Q, the outputs are $0, -4, 5$. The y-intercept for L is $(0, -3)$, and the y-intercept for Q is $(0, -4)$.

Example 6. Use the table to state the domain, range, zeros, and y-intercept.

x	-4	-1	0	2
$f(x)$	0	3	5	0

Answer: Domain: $\{-4, -1, 0, 2\}$. Range: $\{0, 3, 5\}$. Zeros: $x = -4$ and $x = 2$. y-intercept: $(0, 5)$.

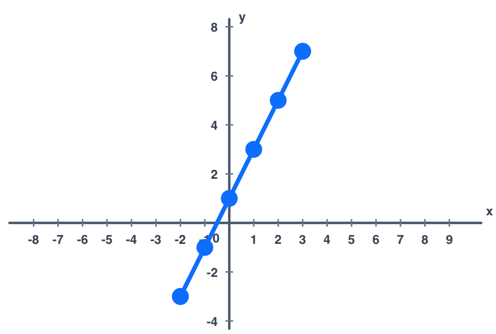
Example 7. Use the graph to name the zeros, x-intercepts, and y-intercept.



Answer: Zeros: $x = -2$ and $x = 3$. x-intercepts: $(-2, 0)$ and $(3, 0)$. y-intercept: $(0, -6)$.

Example 8. Use the table and graph together. State the zero, y-intercept, domain shown, and range shown for the function.

x	-2	-1	0	1	2	3
$f(x)$	-3	-1	1	3	5	7



Answer: The zero is $x = -\frac{1}{2}$, and the y-intercept is $(0, 1)$. The domain shown is $[-2, 3]$. The range shown is $[-3, 7]$.

HW 2 Interval Notation and Graph Features

Use interval notation to describe where a graph increases, decreases, stays constant, and sits above or below the x-axis.

QUICK REFERENCE

- In this section, interval notation is used to describe graph behavior, not just domain and range.

- When reading from a graph, check whether endpoints are open or closed before choosing parentheses or brackets.
- Increasing, decreasing, and constant intervals describe what the outputs do as you move left to right on a graph.
- Domain is the set of x-values shown or allowed. Range is the set of y-values the graph reaches.
- Positive intervals are where the graph is above the x-axis. Negative intervals are where the graph is below the x-axis.
- Average rate of change from $x = a$ to $x = b$ is $\dfrac{f(b) - f(a)}{b - a}$.

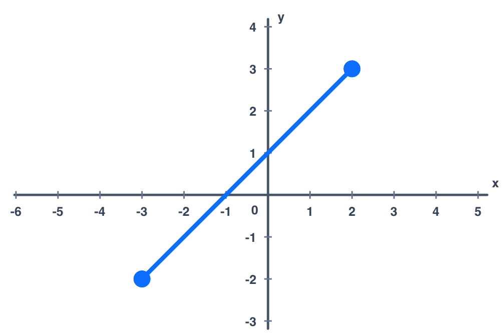
Example 1. Write $x \geq -3$ and $x < 5$ in interval notation, then write $-2 < x \leq 4$ in interval notation.

Answer: $[-3, \infty)$, $(-\infty, 5)$, and $(-2, 4]$.

Example 2. A graph rises on $(-4, -1)$, is flat on $[-1, 2]$, and falls on $(2, 6)$. State the increasing, constant, and decreasing intervals.

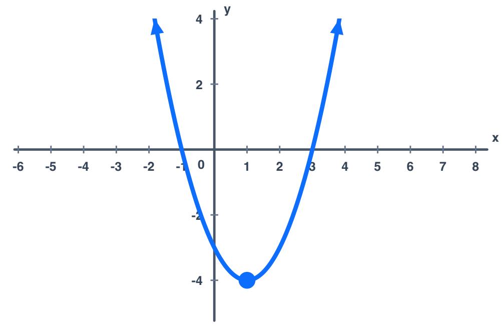
Answer: **Increasing:** $(-4, -1)$. **Constant:** $[-1, 2]$. **Decreasing:** $(2, 6)$.

Example 3. Use the graph of the linear segment to state the domain, range, and whether the function is increasing, decreasing, or constant.



Answer: **Domain:** $[-3, 2]$. **Range:** $[-2, 3]$. **Increasing.**

Example 4. Use the graph of the quadratic to state the domain, range, increasing interval, and decreasing interval.



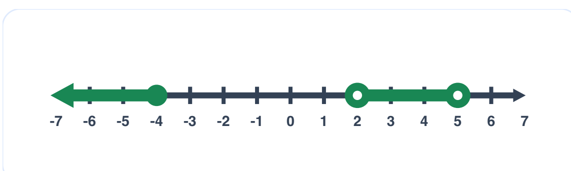
Answer: Domain: all real numbers, $(-\infty, \infty)$. Range: $[-4, \infty)$. Decreasing on $(-\infty, 1)$. Increasing on $(1, \infty)$.

Example 5. A graph crosses the x-axis at $x = -2$ and $x = 3$, is below the x-axis before -2, above between -2 and 3, and below after 3. State the positive and negative intervals.

Answer: Positive on $(-2, 3)$. Negative on $(-\infty, -2) \cup (3, \infty)$.

Example 6. Graph $(-\infty, -4] \cup (2, 5)$ on a number line and describe which endpoints are included.

Answer: The point -4 is included because it has a bracket. The points 2 and 5 are not included because they have parentheses.



Example 7. Use $f(1) = 9$ and $f(7) = -3$ to find the average rate of change from $x = 1$ to $x = 7$.

Answer: $\dfrac{-3 - 9}{7 - 1} = -2$. The output changes by -2 per input unit.

Example 8. Combined graph reading: a function has domain $[-5, 5]$, zeros at $-4, 1, 4$, rises on $(-5, -2)$, falls on $(-2, 3)$, and rises on $(3, 5)$. List zeros, increasing intervals, and decreasing intervals.

Answer: Zeros: $x = -4, 1, 4$. Increasing: $(-5, -2) \cup (3, 5)$. Decreasing: $(-2, 3)$.

HW 3 Parent Functions and Transformation Language

Preview the major function families and the vocabulary used to describe shifts, stretches, compressions, and reflections.

QUICK REFERENCE

- A parent function is the simplest version of a function family. It gives the basic shape before transformations happen.
- Core families here are constant, identity, linear, quadratic, absolute value, square-root, and cube-root functions.
- Common transformations change a parent function in predictable ways.
  - Adding outside the function shifts up or down.
  - Changing inside the input shifts left or right.
  - Multiplying outside by a number stretches, compresses, or reflects the graph across the x-axis.
  - Multiplying the input by a negative reflects the graph across the y-axis.
- We will explore this more later in the course.

Example 1. Match each function to a family: $f(x) = 3$, $g(x) = x$, $h(x) = x^2$, $p(x) = |x|$, $q(x) = \sqrt{x}$.

Answer: Constant, identity/linear, quadratic, absolute value, and square-root.

Example 2. Describe the transformation from $y = x^2$ to $y = (x-4)^2 + 7$.

Answer: Shift right 4 and up 7.

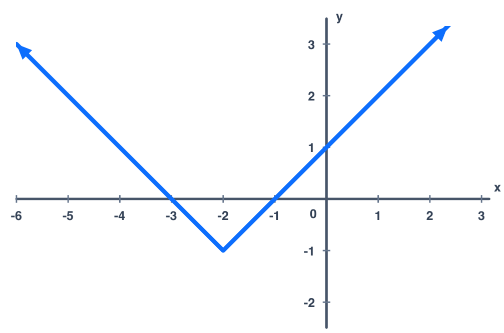
Example 3. Describe the transformation from $y = |x|$ to $y = -2|x+3| - 1$.

Answer: Shift left 3, stretch vertically by 2, reflect across the x-axis, and shift down 1.

Example 4. Describe the transformation from $y = \sqrt{x}$ to $y = \sqrt{x+5} - 2$.

Answer: Shift left 5 and down 2.

Example 5. Use the graph of $y = |x+2| - 1$. Name the parent function, the vertex, and the transformations from the parent graph.



Answer: The parent function is $y = |x|$. The vertex is $(-2, -1)$, so the graph shifts left 2 and down 1.

Module 2: Linear Functions and Models

Students solve linear equations and inequalities, read and write linear functions, work with piecewise linear functions, and use linear models from graphs, tables, formulas, and context.

HW 4 Linear Equations

Solve linear equations, classify edge cases, and rearrange formulas for a specified variable.

QUICK REFERENCE

- Solving an equation means finding the value or values that make the equation true.
- Equivalent equations have the same solution set, even when they look different.
- Use inverse operations to undo what is happening to the variable. Keep both sides balanced.
- Clear fractions by multiplying every term by a common denominator (least common denominator is best), or solve carefully using fraction arithmetic.
- Clear decimals by multiplying by a power of 10 or solve carefully with decimal arithmetic.
- If the variable disappears and the remaining statement is true, the equation has all real numbers as solutions.
- If the variable disappears and the remaining statement is false, the equation has no solution, $\varnothing$.
- A literal equation is a formula with several variables. Solve it by isolating the requested variable.
- Check a solution by substituting it into the original equation and confirming that both sides have the same value.

Example 1. What is unique about the following equations? $2x + 6 = 14$ and $x + 3 = 7$

Answer: **They are equivalent equations because both have the solution $x = 4$.**

Example 2. Solve $5(2n - 3) - 4 = 3n + 16$.

Answer: $n = 5$.

Example 3. Solve $\dfrac{x-2}{3} + \dfrac{x+1}{2} = 5$, then check the solution by substitution.

Answer: $x = \dfrac{31}{5}$. **Check:** $\dfrac{7}{5} + \dfrac{18}{5} = 5$.

Example 4. Solve $\dfrac{a}{2} + \dfrac{5}{6} = \dfrac{2a}{9} + 3$.

Answer: $a = \dfrac{39}{5}$.

Example 5. Solve $0.4p - 1.8 = 2.6$.

Answer: $p = 11$.

Example 6. Solve and classify: $3(x+2) = \frac{1}{2}(6x+12)$.

Answer: All real numbers

Example 7. Solve and classify: $5x - 7 = 5(x+2)$.

Answer: No solution, $\varnothing$.

Example 8. Solve $A = P + Prt$ for r.

Answer: $r = \frac{A-P}{Pt}$.

Example 9. Use the table to solve $2x+1 = -x+10$ numerically. What input makes both expressions equal?

x	2	3	4
$2x+1$	5	7	9
$-x+10$	8	7	6

Answer: At $x = 3$, both expressions equal 7, so the solution is $x = 3$.

HW 5 Linear Inequalities

Solve linear inequalities and represent solution sets with inequalities, intervals, set-builder notation, and number lines.

QUICK REFERENCE

- Solve a linear inequality much like an equation, but keep track of the inequality symbol.
- Reverse the inequality sign when multiplying or dividing both sides by a negative number.
- Be ready to show the same solution in several forms.
  - As an inequality.
  - In interval notation: use parentheses for $<$ or $>$, and brackets for $\leq$ or $\geq$.
  - In set-builder notation, such as $\{x \mid x < 4\}$, which means "the set of all x such that x is less than 4".
  - On a number line: use an open circle for an endpoint not included and a closed circle for an endpoint included. Parentheses and brackets may also be used.

Example 1. Solve $-3(2x-5) > 21$. Write the answer as an inequality, interval notation, set-builder notation, and a number-line graph.

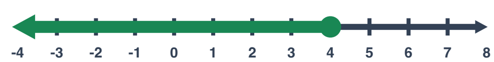
Answer: $x < -1$. **Interval:** $(-\infty, -1)$. **Set-builder:** $\{x \mid x < -1\}$.

Example 2. Graph $-2 < x \leq 4$ on a number-line graph. Write the answer as an inequality, interval notation, and set-builder notation.

Answer: **Inequality:** $-2 < x \leq 4$. **Interval:** $(-2, 4]$. **Set-builder:** $\{x \mid -2 < x \leq 4\}$.

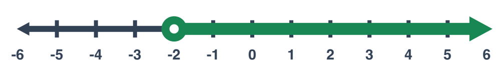
Example 3. Solve $2x - 1 \leq 7$. Write the answer as an inequality, interval notation, set-builder notation, and a number-line graph.

Answer: $x \leq 4$. **Interval:** $(-\infty, 4]$. **Set-builder:** $\{x \mid x \leq 4\}$.



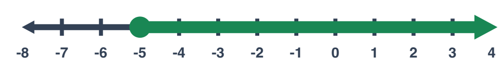
Example 4. Solve $3n + 6 > 0$. Write the answer as an inequality, interval notation, set-builder notation, and a number-line graph.

Answer: $n > -2$. **Interval:** $(-2, \infty)$. **Set-builder:** $\{n \mid n > -2\}$.



Example 5. Write $x \geq -5$ in interval notation and set-builder notation, then show the solution on a number-line graph.

Answer: Inequality: $x \geq -5$. **Interval:** $[-5, \infty)$. **Set-builder:** $\{x \mid x \geq -5\}$.

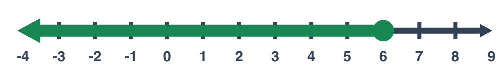


Example 6. Solve $\dfrac{2t-1}{3} \leq 5$. Give inequality, interval, set-builder notation, and a number-line graph.

Answer: $t \leq 8$. **Interval:** $(-\infty, 8]$. **Set-builder:** $\{t \mid t \leq 8\}$.

Example 7. Solve $\dfrac{u-1}{2} + \dfrac{1}{3} \leq \dfrac{2u+5}{6}$. Give inequality, interval, set-builder notation, and a number-line graph.

Answer: $u \leq 6$. **Interval:** $(-\infty, 6]$. **Set-builder:** $\{u \mid u \leq 6\}$.



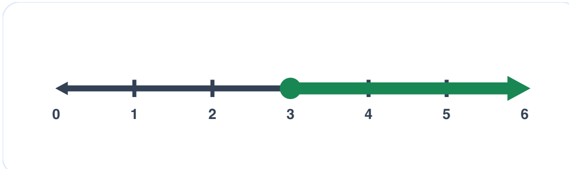
Example 8. Solve $-4x + 7 < 19$. Write the answer as an inequality, interval notation, set-builder notation, and a number-line graph.

Answer: $x > -3$. **Interval:** $(-3, \infty)$. **Set-builder:** $\{x \mid x > -3\}$.

Example 9. Use test values to solve $2x-5\geq 1$. Write the answer as an inequality, interval notation, set-builder notation, and a number-line graph.

Test value	$x=2$	$x=3$	$x=4$
$2x-5$	-1	1	3
Works for $2x-5\geq 1$?	No	Yes	Yes

Answer: $x\geq 3$. **Interval:** $[3,\infty)$. **Set-builder:** $\{x \mid x\geq 3\}$.



HW 6 Linear Functions and Graphs

Connect slope, intercepts, equations, graph behavior, and parallel or perpendicular lines.

QUICK REFERENCE

- Slope is the rate of change: $\dfrac{\text{change in } y}{\text{change in } x}=\dfrac{y_2-y_1}{x_2-x_1}$.
- A linear function is increasing if its slope is positive, decreasing if its slope is negative, and constant if its slope is zero.
- A linear equation can be written in several useful forms.
  - Slope-intercept form is $f(x)=mx+b$ or $y=mx+b$, where m is slope and b is the y-intercept.
  - Point-slope form is $y-y_1=m(x-x_1)$. It is useful when you know a point and a slope.
  - Standard form is $Ax+By=C$, where A and B are not both zero. It is useful for graphing and finding intercepts.
- The notation $f(x)$ names the output for input x, so $f(3)$ means the y-value when $x=3$.
- For horizontal and vertical lines, think HOY VUX.
  - Horizontal lines have slope 0, have equations of the form $y=k$, and are functions.
  - Vertical lines have undefined slope, have equations of the form $x=k$, and are not functions because they fail the vertical line test.
- Parallel lines have the same slope. Perpendicular slopes are opposite reciprocals.
- A linear function is positive where its graph is above the x-axis and negative where it is below the x-axis.
- Comparing two linear functions means finding where their outputs are equal, greater, or less.

Example 1. Find the slope through $(-3,4)$ and $(5,-2)$.

Answer: $m=-\dfrac{3}{4}$.

Example 2. Find the slope through $(-2, 5)$ and $(4, 5)$, then through $(3, -1)$ and $(3, 7)$.

Answer: The first slope is 0, so the line is horizontal. The second slope is undefined, so the line is vertical.

Example 3. Write the line with slope -3 through $(-2, 5)$ in point-slope, slope-intercept, and standard form.

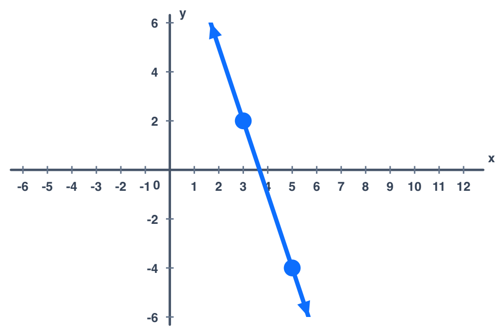
Answer: **Point-slope:** $y - 5 = -3(x + 2)$. **Slope-intercept:** $y = -3x - 1$. **Standard:** $3x + y = -1$.

Example 4. Write the line with slope $\frac{2}{3}$ through $(6, -1)$ in point-slope, slope-intercept, and standard form.

Answer: **Point-slope:** $y + 1 = \frac{2}{3}(x - 6)$. **Slope-intercept:** $y = \frac{2}{3}x - 5$. **Standard:** $2x - 3y = 15$.

Example 5. Graph the line through $(3, 2)$ and $(5, -4)$, then write it in point-slope, slope-intercept, and standard form.

Answer: **Point-slope:** $y - 2 = -3(x - 3)$ **or** $y + 4 = -3(x - 5)$. **Slope-intercept:** $y = -3x + 11$. **Standard:** $3x + y = 11$.



Example 6. A line has x-intercept $(6, 0)$ and y-intercept $(0, -4)$. Write its equation in slope-intercept and standard form.

Answer: **Slope-intercept:** $y = \frac{2}{3}x - 4$. **Standard:** $2x - 3y = 12$.

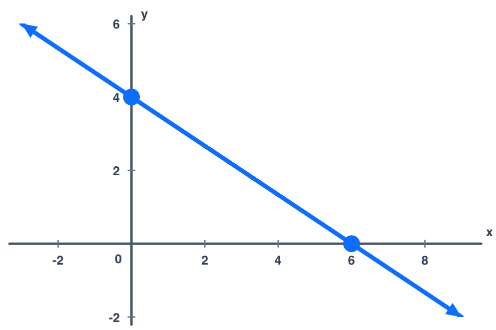
Example 7. Write the horizontal and vertical lines through $(4, -1)$.

Answer: **Horizontal:** $y = -1$. **Vertical:** $x = 4$.

Example 8. Write lines through $(4, -1)$ parallel and perpendicular to $y = 2x + 7$ in slope-intercept form.

Answer: **Parallel:** $y = 2x - 9$. **Perpendicular:** $y = -\frac{1}{2}x + 1$.

Example 9. For $L(x) = -\frac{2}{3}x + 4$, find slope, y-intercept, x-intercept, whether it increases or decreases, and standard form.



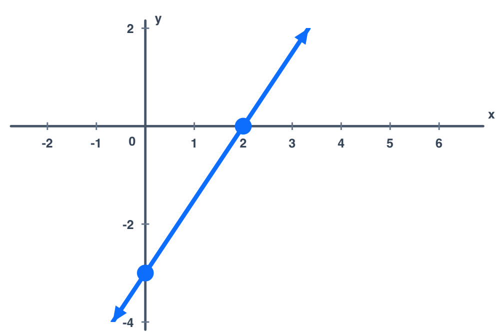
Answer: **Slope:** $-\frac{2}{3}$. **y-intercept:** $(0, 4)$. **x-intercept:** $(6, 0)$. **Decreasing. Standard:** $2x + 3y = 12$.

Example 10. For $f(x) = 2x - 6$, find the zero and solve $f(x) > 0$.

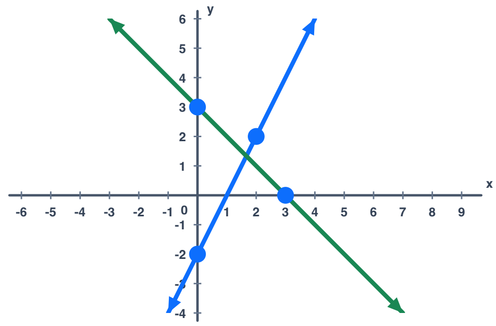
Answer: **Zero:** $x = 3$. **Solution:** $x > 3$, **or** $(3, \infty)$.

Example 11. Graph $y = \frac{3}{2}x - 3$, then state its slope, y-intercept, x-intercept, and standard form.

Answer: **Slope:** $\frac{3}{2}$. **y-intercept:** $(0, -3)$. **x-intercept:** $(2, 0)$. **Standard:** $3x - 2y = 6$.



Example 12. Use the graph to write an equation for each line in slope-intercept form and standard form. Line A has positive slope, and Line B has negative slope.



Answer: Line A: $y = 2x - 2$, $2x - y = 2$. Line B: $y = -x + 3$, $x + y = 3$.

Example 13. Compare $f(x) = 3x - 4$ and $g(x) = x + 2$. Solve $f(x) \leq g(x)$.

Answer: $x \leq 3$.

Example 14. Use the table to write a linear equation in slope-intercept form and function notation, then state the slope and y-intercept.

x	-2	-1	0	1	2
y	-7	-4	-1	2	5

Answer: Slope: 3. y-intercept: $(0, -1)$. Slope-intercept form: $y = 3x - 1$. Function notation: $f(x) = 3x - 1$.

HW 7 Piecewise Linear Functions

Evaluate, graph, and read piecewise linear functions with breakpoints and open or closed endpoints.

QUICK REFERENCE

- A piecewise function uses different rules on different parts of the domain.
- Choose the rule by checking which interval contains the input value.
- Breakpoints are the input values where the rule changes.
- A closed endpoint is included. An open endpoint is not included.
- When reading a piecewise graph, use the filled point at a breakpoint to find the actual function value.

Example 1. Let

$$p(x) = \begin{cases} 2x+1, & \text{if } x < 1 \\ -x+5, & \text{if } x \geq 1 \end{cases}$$

Find $p(-2)$, $p(1)$, and $p(4)$.

Answer: $p(-2) = -3$, $p(1) = 4$, **and** $p(4) = 1$.

Example 2. For the same function, identify the breakpoint and explain which rule is used at the breakpoint.

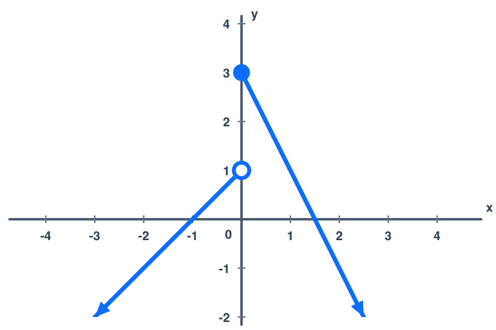
Answer: The breakpoint is $x = 1$. **Use** $-x+5$ **because** $x \geq 1$ **includes** 1.

Example 3. Graph

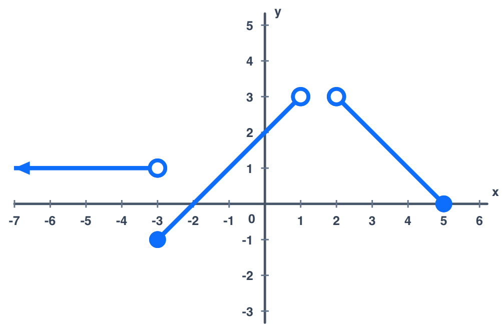
$$p(x) = \begin{cases} x+1, & \text{if } x < 0 \\ -2x+3, & \text{if } x \geq 0 \end{cases}$$

Then find $p(0)$.

Answer: $p(0) = 3$.



Example 4. Use the graph to write the three-piece function $f(x)$, then state the domain.



Answer: $f(x) = \begin{cases} 1, & x < -3 \\ x+2, & -3 \le x < 1 \\ -x+5, & 2 < x \le 5 \end{cases}$. **Domain:** $(-\infty, 1) \cup (2, 5]$.

HW 8 Linear Modeling and Data

Use linear models, break-even comparisons, and scatterplot patterns to make and interpret predictions.

QUICK REFERENCE

- A linear model can be written as $f(x) = mx + b$, where slope is the rate of change and the y-intercept is the starting value.
- The independent variable is the input. The dependent variable is the output that changes in response.
- Domain and range should make sense in context, not just in algebra.
- A model can be used forward to find an output from an input or backward to find the input that gives a known output.
- Break-even means two models have the same output for the same input.
- A scatterplot and its correlation coefficient describe the direction and strength of a linear relationship.
  - An upward pattern shows positive correlation, a downward pattern shows negative correlation, and no clear pattern shows little correlation.
  - A correlation coefficient near 1 or -1 means the points are close to a line. A value near 0 means a weak linear pattern.
- A line of best fit gives an approximate model for prediction. Predictions should be interpreted with units and context.

Example 1. A phone plan costs $\$35$ plus $\$0.08$ per minute. Write a model, find the cost for 450 minutes, and find the number of minutes that gives a bill of $\$83$.

Answer: $C(m) = 35 + 0.08m$. **Cost for 450 minutes: $\$71$. A bill of $\$83$ corresponds to 600 minutes.**

Example 2. Plan A costs $\$20$ plus $\$0.05$ per page. Plan B costs $\$8$ plus $\$0.11$ per page. Find the break-even point and say when Plan A is cheaper.

Answer: **Break-even is 200 pages. Plan A is cheaper for more than 200 pages.**

Example 3. A gym charges a $\$15$ signup fee plus $\$22$ per month. Write a model for the total cost during the first year, then state a realistic domain and range.

Answer: $C = 15 + 22m$. **Domain:** $0 \le m \le 12$. **Range:** $15 \le C \le 279$.

Example 4. A scatterplot rises from left to right and the points are close to a line. Describe the correlation.

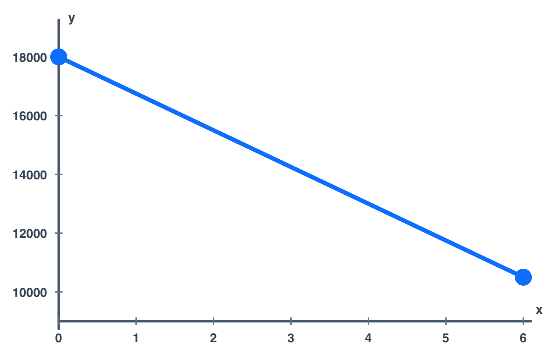
Answer: **Strong positive correlation.**

Example 5. A plant is 18 inches tall and grows about 2.5 inches each week. Use the model $H = 2.5w + 18$ to predict the height after 6 weeks and interpret the slope.

Answer: $H(6) = 33$ **inches. The plant grows about 2.5 inches per week.**

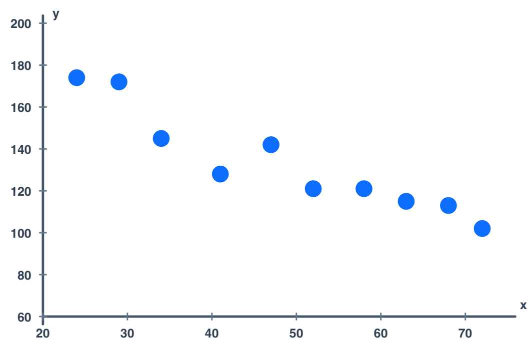
Example 6. A car is worth $\$18{,}000$ now and loses $\$1{,}250$ of value each year. Write a model, name the independent and dependent variables, give a realistic domain for the first six years, and graph the model on that domain.

Answer: $V(t) = 18000 - 1250t$. **Time t is independent; value V is dependent. Domain:** $0 \le t \le 6$. **Range:** $10500 \le V \le 18000$.



Example 7. A utility company samples heating bills from different homes during winter. Use the data table and scatterplot to describe the correlation between average outdoor temperature and monthly heating cost. Then find a line using the first and last points, find the linear regression equation, and use both equations to predict the heating cost when the average outdoor temperature is $52°\mathrm{F}$. Use exact values for the first/last-point model; round the regression equation to the nearest hundredth and predictions to the nearest dollar.

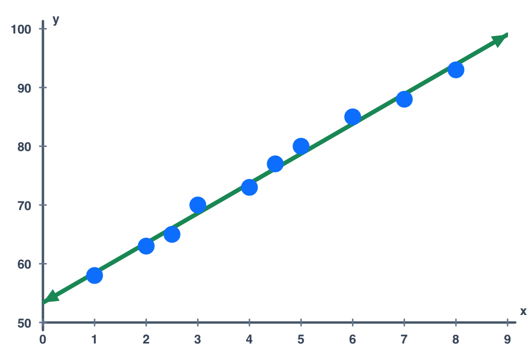
Temperature (°F)	24	29	34	41	47	52	58	63	68	72
Heating cost	\$174	\$172	\$145	\$128	\$142	\$121	\$121	\$115	\$113	\$102



Answer: Strong negative correlation, with $r \approx -0.938$. First/last model: $C = -1.5t + 210$, prediction $C(52) = \$132$. Regression model: $C = -1.38t + 200.84$, prediction $C(52) \approx \$129$.

Example 8. Use the study-hours data to find a line of best fit, describe the correlation, and predict the test score for 5.5 hours of studying. Round the slope and y-intercept to the nearest hundredth, the correlation coefficient to the nearest thousandth, and the predicted score to the nearest whole point.

Study hours	1	2	2.5	3	4	4.5	5	6	7	8
Test score	58	63	65	70	73	77	80	85	88	93



Answer: A line of best fit is approximately $S = 5.07h + 53.38$, where h is study hours and S is test score. The correlation is strong and positive, with $r \approx 0.996$. For 5.5 hours, $S(5.5) \approx 5.07(5.5) + 53.38 \approx 81.3$, so the model predicts about 81 points.

Module 3: Quadratic Equations and Quadratic Functions

Students extend function analysis from lines to curves through factoring, quadratic solving, complex numbers, graphing, inequalities, and applications.

HW 9 Factoring Foundations

Review the factoring patterns students need for quadratic and polynomial work.

QUICK REFERENCE

- Factoring rewrites an expression as multiplication so the pieces are easier to use later.
- Always check for a greatest common factor first.
- After checking for a GCF, choose the factoring pattern that matches the expression.
  - For trinomials in the form $ax^2 + bx + c$, look for two factors that multiply to the constant term and add to the middle coefficient, or use grouping when $a \neq 1$.
  - For four-term polynomials, use grouping to factor two pairs of terms, then factor the common binomial.
  - "Bottoms Up" can be used to factor trinomials with $a \neq 1$. Take care of any GCF first.
  - Difference of squares follows $a^2 - b^2 = (a - b)(a + b)$.
  - Sum and difference of cubes use special patterns. The SOAP shortcut helps with signs: Same sign, Opposite sign, Always Positive. The formulas are $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
  - Perfect-square trinomials follow $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$.
- If no integer factoring pattern works, say the polynomial is prime.
- Factoring completely means none of the remaining factors can be factored further over the number system being used.

Example 1. Factor: $6x^3 - 24x^2$.

Answer: $6x^2(x - 4)$.

Example 2. Factor: $x^2 - 7x + 12$.

Answer: $(x - 3)(x - 4)$.

Example 3. Factor completely: $3n^2 + 18n + 24$.

Answer: $3(n + 2)(n + 4)$.

Example 4. Factor: $y^2 + y - 12$.

Answer: $(y+4)(y-3)$.

Example 5. Factor: $x^2 - 2x - 15$.

Answer: $(x-5)(x+3)$.

Example 6. Factor $x^2 + 5x + 11$.

Answer: Prime

Example 7. Factor by grouping: $x^3 + 2x^2 + 3x + 6$.

Answer: $(x+2)(x^2+3)$.

Example 8. Factor: $2x^2 + 7x + 3$.

Answer: $(2x+1)(x+3)$.

Example 9. Factor: $6x^2 + x - 2$.

Answer: $(3x+2)(2x-1)$.

Example 10. Factor: $a^2 + 10a + 25$.

Answer: $(a+5)^2$.

Example 11. Factor: $4n^2 - 20n + 25$.

Answer: $(2n-5)^2$.

Example 12. Factor: $9x^2 - 25$.

Answer: $(3x-5)(3x+5)$.

Example 13. Factor $x^4 - 81$ completely.

Answer: $(x-3)(x+3)(x^2+9)$.

Example 14. Factor $2x^3 - 18x$ completely.

Answer: $2x(x-3)(x+3)$.

Example 15. Factor: $8b^3 + 27$.

Answer: $(2b+3)(4b^2-6b+9)$.

Example 16. Factor: $27y^3 - 8$.

Answer: $(3y-2)(9y^2+6y+4)$.

HW 10 Solving Quadratics by Factoring

Use factoring and the zero product property to find zeros and repeated solutions.

QUICK REFERENCE

- Set the quadratic equal to 0 before using factoring to solve.
- Zero product property: if $uv = 0$, then $u = 0$ or $v = 0$.
- Solutions, zeros, and x-intercepts describe the same values in different settings.
  - A zero is an input that makes the output equal 0, so solving $f(x) = 0$ finds the zeros.
  - A real solution or real zero becomes an x-intercept when the equation is written as a function.
  - A repeated solution means the graph touches the x-axis instead of crossing it there.
- Some quadratics do not factor nicely, so more solving techniques will be shown later.

Example 1. Solve: $(x-4)(x+2) = 0$.

Answer: $x = 4$ or $x = -2$.

Example 2. Solve by factoring: $n^2 - 3n = 0$.

Answer: $n = 0$ or $n = 3$.

Example 3. Solve by factoring: $x^2 - 7x + 12 = 0$.

Answer: $x = 3$ or $x = 4$.

Example 4. Solve $x^2 + 5x = 14$ by moving everything to one side first.

Answer: $x = -7$ or $x = 2$.

Example 5. Solve $2(x+1)(x-3) = x(x-5) - 6$ by expanding and moving all terms to one side before factoring.

Answer: $x = -1$ or $x = 0$.

Example 6. Solve by factoring: $2p^2 - 5p - 3 = 0$.

Answer: $p = 3$ or $p = -\frac{1}{2}$.

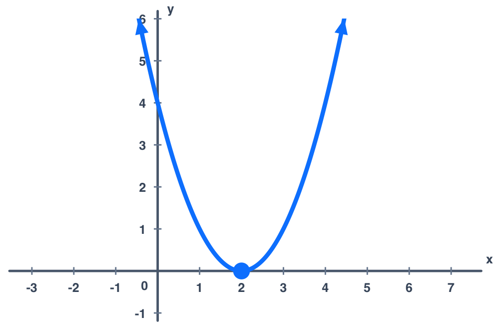
Example 7. For $f(x) = (x+2)(x-5)$, state the zeros and x-intercepts.

Answer: Zeros: $x = -2$ and $x = 5$. x-intercepts: $(-2, 0)$ and $(5, 0)$.

Example 8. Solve and identify the repeated solution: $x^2 - 6x + 9 = 0$.

Answer: $x = 3$, repeated solution.

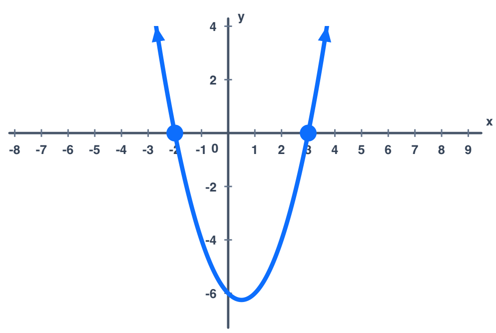
Example 9. Use the graph of $f(x) = (x-2)^2$ to state the zero, x-intercept, and whether the solution is repeated.



Answer: Zero: $x = 2$. x-intercept: $(2, 0)$. Repeated solution: yes.

Example 10. Use the table and graph of $f(x) = x^2 - x - 6$ to identify the zeros before solving by factoring.

x	-3	-2	-1	0	1	2	3
$f(x)$	6	0	-4	-6	-6	-4	0



Answer: Zeros: $x = -2$ and $x = 3$. x-intercepts: $(-2, 0)$ and $(3, 0)$. Solutions: $x = -2$ and $x = 3$.

Example 11. Solve $m^3 - 4m = 0$.

Answer: $m = -2, 0, 2$.

HW 11 Square Roots, Radicals, and Complex Number Basics

Connect square roots, simplified radicals, and complex-number arithmetic.

QUICK REFERENCE

- Square roots solve equations of the form $x^2 = k$, giving $x = \pm\sqrt{k}$ when $k > 0$.

- $\sqrt{49} = 7$, but $x^2 = 49$ gives $x = \pm 7$.
- Simplify square roots by pulling out perfect-square factors and cube roots by pulling out perfect-cube factors. Cube roots of negative numbers are real and negative.
- The imaginary unit satisfies $i^2 = -1$, so negative square roots use i: $\sqrt{-a} = i\sqrt{a}$.
- Powers of i repeat every four powers: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$.
- Complex numbers combine real and imaginary parts: $a + bi$.
- Use the operation to decide how to work with complex numbers.
  - Add and subtract by combining real parts with real parts and imaginary parts with imaginary parts.
  - Multiply by distributing and replacing i^2 with -1.
  - When dividing, use complex conjugates to remove i from the denominator.

Example 1. Simplify $\sqrt{72}$ and $\sqrt{45}$.

Answer: $\sqrt{72} = 6\sqrt{2}$. $\sqrt{45} = 3\sqrt{5}$.

Example 2. Simplify $\sqrt[3]{54}$ and $\sqrt[3]{-128}$.

Answer: $\sqrt[3]{54} = 3\sqrt[3]{2}$. $\sqrt[3]{-128} = -4\sqrt[3]{2}$.

Example 3. Simplify $\sqrt{-48}$ and $\sqrt{-75}$.

Answer: $\sqrt{-48} = 4i\sqrt{3}$. $\sqrt{-75} = 5i\sqrt{3}$.

Example 4. Use the table to decide which square roots are real numbers and which require i.

Expression	$\sqrt{49}$	$\sqrt{0}$	$\sqrt{-9}$	$\sqrt{-20}$
Simplified form				
Type				

Answer: **Simplified forms: 7, 0, $3i$, $2i\sqrt{5}$. Types: Real, Real, Imaginary, Imaginary.**

Example 5. Solve $y^2 = 64$.

Answer: $y = -8$ or $y = 8$.

Example 6. Solve $r^2 = -36$.

Answer: $r = -6i$ or $r = 6i$.

Example 7. Solve $3(t-2)^2 = 75$.

Answer: $t = -3$ or $t = 7$.

Example 8. Simplify i^{27}.

Answer: $-i$.

Example 9. Add and subtract: $(5+2i)+(3-7i)$ and $(5+2i)-(3-7i)$.

Answer: **Sum:** $8-5i$. **Difference:** $2+9i$.

Example 10. Compute $(3-2i)(4+i)$.

Answer: $14-5i$.

Example 11. Name the complex conjugate of $6-5i$, then multiply the number by its conjugate.

Answer: **Conjugate:** $6+5i$. **Product:** 61.

Example 12. Divide $\dfrac{5}{2-i}$.

Answer: $2+i$.

HW 12 Quadratic Formula, Completing the Square, and Discriminant

Use the quadratic formula, completing the square, and the discriminant to classify and solve quadratic equations.

QUICK REFERENCE

- Write the equation in standard form $ax^2+bx+c=0$ before using the quadratic formula.
- Quadratic formula: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.
- Completing the square rewrites a quadratic into a perfect-square form, such as $(x-h)^2=k$, so square roots can be used.
- Completing the square is not always the fastest method, but it is useful for graphing and understanding vertex form $f(x)=a(x-h)^2+k$, which will be covered soon.
- The discriminant is $D=b^2-4ac$. It tells what kind of solutions to expect before solving.
- The sign of the discriminant identifies the solution and x-intercept cases.
  - If $D>0$, there are two real solutions and two x-intercepts.
  - If $D=0$, there is one repeated real solution and one x-intercept.
  - If $D<0$, there are two complex solutions and no real x-intercepts.
- Exact radical answers should stay exact unless a decimal is requested.
- Complex answers should be written in $a+bi$ form when the discriminant is negative.
- The quadratic formula works even when factoring does not.

Example 1. For $4x^2-7x+3=0$, identify a, b, and c.

Answer: $a=4$, $b=-7$, $c=3$.

Example 2. Find the discriminant of $x^2-8x+16=0$ and state the number of real solutions.

Answer: $D=0$. **One repeated real solution.**

Example 3. Complete the discriminant table and use it to predict the number of real x-intercepts before solving.

Equation	$D = b^2 - 4ac$	Real x-intercepts
$x^2 - 6x + 9 = 0$		
$x^2 - 5x + 4 = 0$		
$x^2 + 2x + 5 = 0$		

Answer: $D = 0$, one repeated x-intercept. $D = 9$, two x-intercepts. $D = -16$, no real x-intercepts.

Example 4. Use the quadratic formula to solve $2n^2 - 3n - 2 = 0$.

Answer: $n = 2$ or $n = -\frac{1}{2}$.

Example 5. Use the quadratic formula to solve $x^2 - 4x + 4 = 0$, and identify whether the solution is repeated.

Answer: $x = 2$, repeated solution.

Example 6. Use the quadratic formula to solve $x^2 + 2x + 5 = 0$, and write answers in $a + bi$ form.

Answer: $x = -1 \pm 2i$.

Example 7. Solve $2p^2 - 4p - 1 = 0$ by completing the square, and leave exact irrational answers.

Answer: $p = \frac{2 \pm \sqrt{6}}{2}$.

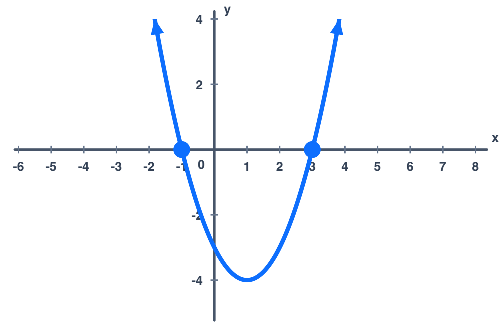
Example 8. Solve $x^2 + x - 1 = 0$ by completing the square and by the quadratic formula.

Answer: Both methods give $x = \frac{-1 \pm \sqrt{5}}{2}$.

Example 9. Use the discriminant to classify and solve $3x^2 + 2x - 1 = 0$.

Answer: $D = 16$. Two real solutions: $x = -1$ and $x = \frac{1}{3}$.

Example 10. Use the graph to decide whether the discriminant of the quadratic is positive, zero, or negative.



Answer: **Positive discriminant. The graph has two real x-intercepts.**

HW 13 Quadratic Graphs and Transformations

Read quadratic graphs through vertices, intercepts, transformations, intervals, and major edge cases.

QUICK REFERENCE

- A quadratic graph is a parabola. The vertex is the highest or lowest point.
- Vertex form $f(x) = a(x-h)^2 + k$ shows the vertex and uses the same transformation language introduced with parent functions.
  - The vertex is (h, k).
  - The value of h shifts the graph left or right, and k shifts it up or down.
  - The value of a controls vertical stretch, compression, or reflection.
  - If $a > 0$, the parabola opens up and has a minimum. If $a < 0$, it opens down and has a maximum.
- The axis of symmetry is the vertical line through the vertex.
- For standard form $f(x) = ax^2 + bx + c$, the y-intercept is $(0, c)$, and the axis of symmetry is $x = -\dfrac{b}{2a}$.
- Domain is usually all real numbers, $(-\infty, \infty)$. Range depends on the vertex and opening direction.
- Zeros, x-intercepts, and the discriminant connect the graph and equation.
- Completing the square can rewrite standard form into vertex form $f(x) = a(x-h)^2 + k$, which makes the vertex and transformations easier to read.
- A point table can help graph transformed parabolas, especially when the vertex or intercepts are not integers.
- Piecewise graphs with quadratic pieces still require careful endpoint and interval reading.

Example 1. For $q(x) = -2(x-3)^2 + 8$, state vertex, axis, opening direction, domain, range, and increasing/decreasing intervals.

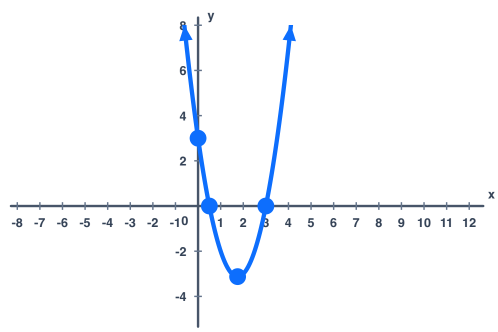
Answer: **Vertex:** $(3, 8)$. **Axis:** $x = 3$. **Opens down. Domain: all real numbers,** $(-\infty, \infty)$. **Range:** $(-\infty, 8]$. **Increasing:** $(-\infty, 3)$. **Decreasing:** $(3, \infty)$.

Example 2. For $f(x) = x^2 - 4x - 5$, find the vertex, x-intercepts, y-intercept, and axis of symmetry.

Answer: **Vertex:** $(2, -9)$. **x-intercepts:** $(-1, 0)$ **and** $(5, 0)$. **y-intercept:** $(0, -5)$. **Axis:** $x = 2$.

Example 3. Rewrite $g(x) = 2x^2 - 7x + 3$ in vertex form by completing the square. Then graph it and state the vertex, axis of symmetry, zeros, y-intercept, opening direction, and range.

Answer: **Vertex form:** $g(x) = 2\left(x - \frac{7}{4}\right)^2 - \frac{25}{8}$. **Vertex:** $\left(\frac{7}{4}, -\frac{25}{8}\right)$. **Axis:** $x = \frac{7}{4}$. **Zeros:** $x = \frac{1}{2}$ **and** $x = 3$. **y-intercept:** $(0, 3)$. **Opens up. Range:** $\left[-\frac{25}{8}, \infty\right)$.

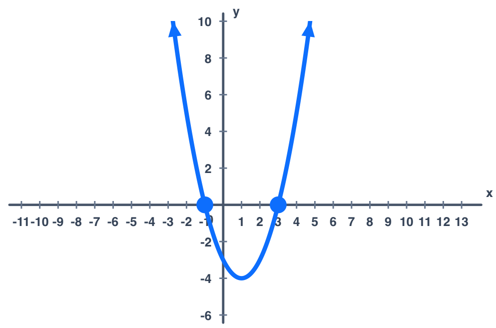


Example 4. Complete the value table for $q(x) = 2(x-1)^2 - 3$, then identify the vertex, axis of symmetry, and range.

x	-1	0	1	2	3
$q(x)$					

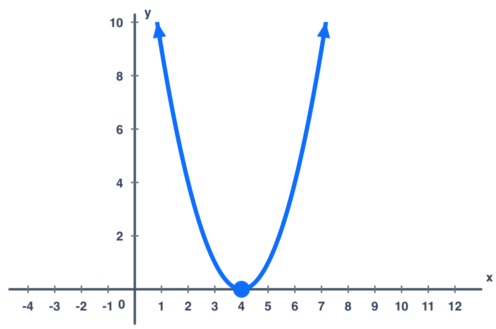
Answer: **Outputs:** $5, -1, -3, -1, 5$. **Vertex:** $(1, -3)$. **Axis:** $x = 1$. **Range:** $[-3, \infty)$.

Example 5. Use the graph of $y = (x+1)(x-3)$ to state the zeros and x-intercepts.



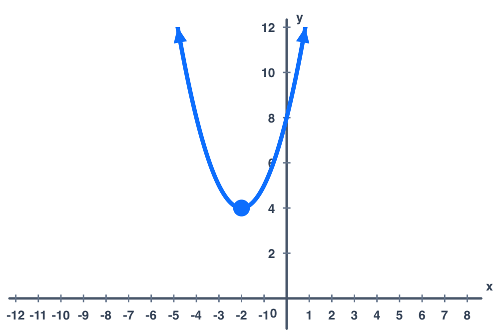
Answer: Zeros: $x = -1$ and $x = 3$. x-intercepts: $(-1, 0)$ and $(3, 0)$.

Example 6. Use the graph of $g(x) = (x-4)^2$. Give the zero and graph behavior at the x-axis.



Answer: Zero: $x = 4$. The graph touches the x-axis at $(4, 0)$.

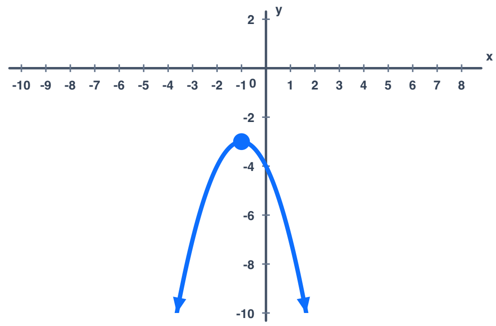
Example 7. Use the graph of $h(x) = x^2 + 4x + 8$. Decide whether the graph has no real zeros, one real zero, or two real zeros.



Answer: No real zeros. The graph does not touch or cross the x-axis.

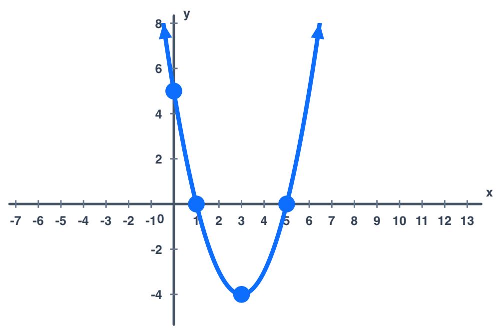
Example 8. Graph $p(x) = -(x+1)^2 - 3$, then state the vertex, range, and whether there are real zeros.

Answer: Vertex: $(-1, -3)$. **Range:** $(-\infty, -3]$. **No real zeros.**

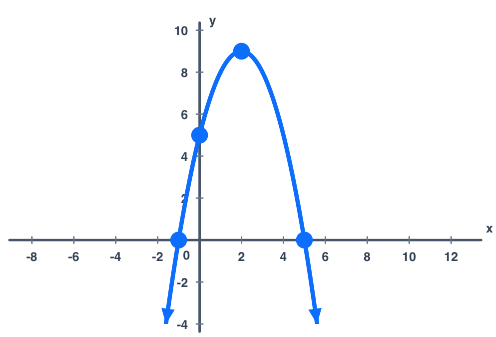


Example 9. Rewrite $f(x) = x^2 - 6x + 5$ in vertex form, graph it, then state the vertex, axis of symmetry, opening direction, and range.

Answer: $f(x) = (x-3)^2 - 4$. **Vertex:** $(3, -4)$. **Axis:** $x = 3$. **Opens up. Range:** $[-4, \infty)$.

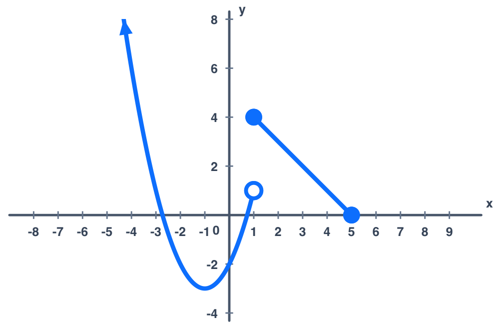


Example 10. Use the graph of $f(x) = -(x-2)^2 + 9$. State intercepts, vertex, domain, range, increasing/decreasing intervals, and where $f(x) > 0$.



Answer: Vertex: $(2, 9)$. **x-intercepts:** $(-1, 0)$, $(5, 0)$. **y-intercept:** $(0, 5)$. **Domain: all real numbers,** $(-\infty, \infty)$. **Range:** $(-\infty, 9]$. **Increasing:** $(-\infty, 2)$. **Decreasing:** $(2, \infty)$. **Positive on** $(-1, 5)$.

Example 11. Use the mixed piecewise graph. State the domain, range, $f(1)$, the open endpoint, and which part is quadratic.



Answer: **Domain:** $(-\infty, 5]$. **Range:** $[-3, \infty)$. $f(1) = 4$. **Open endpoint:** $(1, 1)$. **Quadratic part: the left curved piece.**

HW 14 Quadratic Inequalities

Solve quadratic inequalities with graphs, sign intervals, and endpoint decisions.

QUICK REFERENCE

- A quadratic inequality asks where a parabola is above or below a boundary, usually the x-axis.
- When solving algebraically, move everything to one side so you can compare the quadratic to 0.
- Use the zeros and the intervals they create to determine the sign of a quadratic.
  - The zeros are the critical values. They split the number line into intervals to check.
  - A sign table keeps the intervals and test values organized.
  - Test one value in each interval, or use the graph, to decide where the quadratic is positive or negative.
  - The opening direction and zeros can often tell the sign pattern quickly: opening up is positive outside the zeros and negative between; opening down is the reverse.
- Use brackets when the inequality includes equality, such as $\leq$ or $\geq$. Use parentheses when it does not.
- Write the final answer in interval notation, or other appropriate notations mentioned earlier in the course, and show it on a number line when asked.
- When comparing two functions, find where they meet, then decide which graph is above or below.

Example 1. Solve $x^2 - x - 6 \leq 0$.

Answer: $[-2, 3]$.

Example 2. Use a sign table to help solve $2x^2 - 5x - 3 \leq 0$.

Interval			
Test value			
Sign			

Answer: $\left[-\frac{1}{2}, 3\right]$.

Interval	$\left(-\infty, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, 3\right)$	$(3, \infty)$
Test value	-1	0	4
Sign	Positive	Negative	Positive

Example 3. Solve $n^2 - 9 > 0$.

Answer: $(-\infty, -3) \cup (3, \infty)$.

Example 4. Solve $-x^2 + 4x + 5 \geq 0$.

Answer: $[-1, 5]$.

Example 5. A parabola opens up and has x-intercepts -4 and 2. Where is it negative?

Answer: $(-4, 2)$.

Example 6. Solve $t^2 + 4 > 0$.

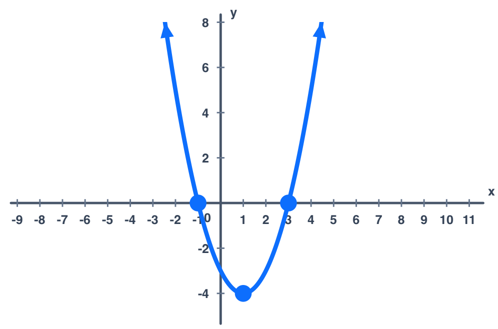
Answer: All real numbers, $(-\infty, \infty)$.

Example 7. Solve $t^2 + 4 < 0$.

Answer: No real solution, $\varnothing$.

Example 8. Graph $y = x^2 - 2x - 3$, then solve $x^2 - 2x - 3 \leq 0$.

Answer: $[-1, 3]$.



Example 9. Use a sign table to solve $(x + 2)(x - 3) > 0$, then graph the solution on a number line.

Interval			
Test value			
Sign of product			

Answer: **Solution:** $(-\infty, -2) \cup (3, \infty)$.

Interval	$(-\infty, -2)$	$(-2, 3)$	$(3, \infty)$
Test value	-3	0	4
Sign of product	Positive	Negative	Positive

Example 10. Solve $f(x) \leq g(x)$ for $f(x) = x^2 - 4$ and $g(x) = 2x - 1$.

Answer: $[-1, 3]$.

HW 15 Quadratic Applications

Interpret projectile, area, revenue, vertex, and intercept information in context.

QUICK REFERENCE

- In applications, the vertex often gives the maximum or minimum value.

- Projectile models usually open down, so the vertex gives maximum height.
- Intercepts can represent starting value, landing time, break-even points, or zero output.
- Area, revenue, profit, and cost models often ask which input gives the largest or smallest output.
- A quadratic regression model is an approximate model from data, so use units and interpret predictions in context.
- Revenue and profit are different models. Revenue is money coming in, and profit accounts for costs using $P(x) = R(x) - C(x)$.
- Answers need units and a sentence explaining what the value means in context.

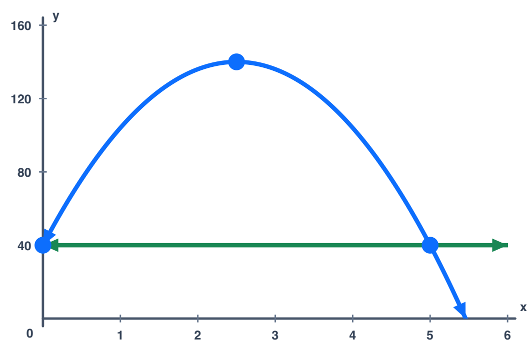
Example 1. A ball has height $h(t) = -16t^2 + 64t + 5$. Find the maximum height and when it occurs.

Answer: **Maximum height: 69 feet at $t = 2$ seconds.**

Example 2. For $h(t) = -16t^2 + 48t$, find when the object hits the ground after launch.

Answer: **The object hits the ground 3 seconds after launch.**

Example 3. A model $h(t) = -16t^2 + 80t + 40$ gives a ball's height. During what time interval is the ball above the 40-foot rooftop?



Answer: **The ball is above the rooftop for $0 < t < 5$ seconds.**

Example 4. Use the table for $h(t) = -16t^2 + 64t + 5$. Estimate the maximum height and explain how the table supports the vertex result.

t seconds	0	1	2	3	4
$h(t)$ feet	5	53	69	53	5

Answer: **Maximum height: 69 feet at $t = 2$ seconds. The table is symmetric around $t = 2$.**

Example 5. Revenue is $R(x) = -2x^2 + 80x$. Find the value of x that maximizes revenue and the maximum revenue.

Answer: **$x = 20$. Maximum revenue: 800.**

Example 6. A farmer has 30 feet of fencing for three sides of a rectangular pen placed against a barn. Let x be the length of each side perpendicular to the barn. Write the area model and find the maximum area.

Answer: $A(x) = x(30 - 2x)$. **Maximum area:** 112.5 **square feet when** $x = 7.5$ **feet.**

Example 7. A company's cost model is $C(x) = 0.02x^2 - 4x + 350$, where x is the number of items produced. Find the number of items that minimizes cost and the minimum cost.

Answer: **Minimum cost:** $\$150$ **when** $x = 100$ **items.**

Example 8. A product has revenue $R(x) = -4x^2 + 160x$ and cost $C(x) = 40x + 500$. Write $P(x) = R(x) - C(x)$. Then compare the revenue maximum and profit maximum, and find the break-even values.

Answer: $P(x) = -4x^2 + 120x - 500$. **Revenue maximum:** 1600 **at** $x = 20$. **Profit maximum:** 400 **at** $x = 15$. **Break-even values:** $x = 5$ **and** $x = 25$.

Example 9. A traffic study records stopping distance at different speeds. Use quadratic regression on the table to find a model $d(v)$, where v is speed in mph and $d(v)$ is stopping distance in feet. Round the displayed model coefficients to the nearest thousandth, but use the unrounded regression values for predictions. Then predict the stopping distance at 45 mph and find the speed that gives a stopping distance of 200 feet, rounding both results to the nearest whole number.

Speed v (mph)	20	30	40	50	60
Stopping distance $d(v)$ (feet)	53	88	140	190	262

Answer: $d(v) \approx 0.051v^2 + 1.086v + 10.600$. **At** 45 **mph, the predicted stopping distance is about** 164 **feet. A stopping distance of** 200 **feet corresponds to about** 51 **mph.**

Module 4: Absolute Value and Radicals

Students solve and graph absolute value and radical relationships while watching domains, endpoints, transformations, and extraneous solutions.

HW 16 Absolute Value Equations and Inequalities

Solve absolute value equations and inequalities using distance and two-case reasoning.

QUICK REFERENCE

- Absolute value measures distance from zero, so an absolute value result cannot be negative.
- Isolate the absolute value expression before splitting into cases or writing a compound inequality.

- After isolating the absolute value, use the case that matches the other side.
  - If $|A| = b$ with $b > 0$, solve $A = b$ or $A = -b$.
  - If $|A| = 0$, solve $A = 0$.
  - If $|A|$ equals a negative number, there is no solution, $\varnothing$.
- Absolute value inequalities have several common solution patterns.
  - Less-than inequalities usually make one inside interval.
  - Greater-than inequalities usually make two outside intervals.
  - Some inequalities are always true or never true because absolute value is never negative.
- Distance language helps explain why the solution is inside or outside a range.
- Be ready to write solutions as inequalities, interval notation, and number-line graphs.

Example 1. Solve $|2x - 3| = 7$.

Answer: $x = 5$ **or** $x = -2$.

Example 2. Solve $2|n - 3| - 5 = 9$.

Answer: $n = 10$ **or** $n = -4$.

Example 3. Solve $|x + 4| = 0$ and $|x - 5| = -2$.

Answer: $|x + 4| = 0$ **gives** $x = -4$. $|x - 5| = -2$**: no solution,** $\varnothing$.

Example 4. Solve $|x - 2| < 5$.

Answer: $-3 < x < 7$, **or** $(-3, 7)$.

Example 5. Solve $3|2r + 1| + 4 \leq 16$.

Answer: $-\frac{5}{2} \leq r \leq \frac{3}{2}$, **or** $\left[-\frac{5}{2}, \frac{3}{2}\right]$.

Example 6. Solve $|x + 1| \geq 3$.

Answer: $x \leq -4$ **or** $x \geq 2$, **so** $(-\infty, -4] \cup [2, \infty)$.

Example 7. Write $|t - 6| \leq 2$ as a distance statement and solve.

Answer: t **is within** 2 **units of** 6. **Solution:** $[4, 8]$.

Example 8. Solve $|3x - 6| \geq 0$.

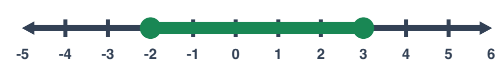
Answer: All real numbers, because an absolute value is always greater than or equal to 0.

Example 9. Solve $|x + 2| < -4$.

Answer: No solution, $\varnothing$.

Example 10. Solve $|2x-1| \leq 5$. Give interval notation and show the solution on a number line.

Answer: $[-2, 3]$.



Example 11. Solve $|x-3| > 4$ and show the solution on a number line.

Answer: **The expression means the distance from 3 is more than 4. So $x < -1$ or $x > 7$, which is** $(-\infty, -1) \cup (7, \infty)$.

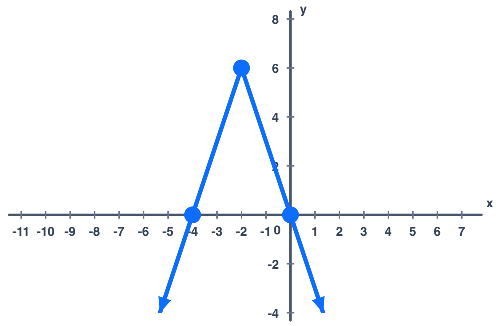
HW 17 Absolute Value Functions and Graphs

Read absolute value graphs through vertex, transformations, intercepts, intervals, and graphical solving.

QUICK REFERENCE

- Absolute value graphs have a vertex and two straight sides.
- The form $f(x) = a|x-h| + k$, which may also be written with $y =$, shows the graph's key features and uses the same transformation language introduced with parent functions.
  - The vertex is (h, k).
  - The value of h shifts the graph left or right, and k shifts it up or down.
  - The value of a controls stretch, compression, or reflection. If $a > 0$, the graph opens up; if $a < 0$, it opens down.
  - Domain is usually all real numbers, $(-\infty, \infty)$. Range depends on the vertex and opening direction.

Example 1. Use the graph of $g(x) = -3|x+2| + 6$. State vertex, direction, intercepts, domain, range, and increasing/decreasing intervals.



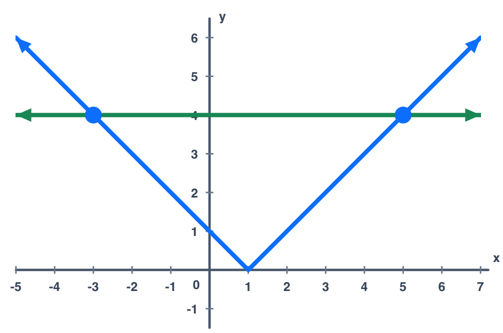
Answer: Vertex: $(-2, 6)$. Opens down. x-intercepts: $(-4, 0)$, $(0, 0)$. y-intercept: $(0, 0)$. Domain: all real numbers, $(-\infty, \infty)$. Range: $(-\infty, 6]$. Increasing: $(-\infty, -2)$. Decreasing: $(-2, \infty)$.

Example 2. For $f(x) = |x-4| - 3$, state the vertex, range, and zeros.

Answer: Vertex: $(4, -3)$. Range: $[-3, \infty)$. Zeros: $x = 1$ and $x = 7$.

Example 3. Solve graphically/algebraically: $|x-1| = 4$.

Answer: $x = -3$ or $x = 5$.

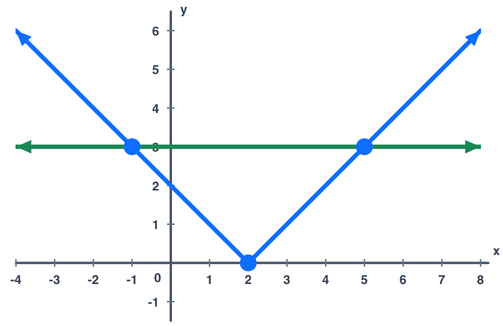


Example 4. For $a(x) = 2|x+1| - 5$, state where the function is decreasing and increasing.

Answer: Decreasing on $(-\infty, -1)$. Increasing on $(-1, \infty)$.

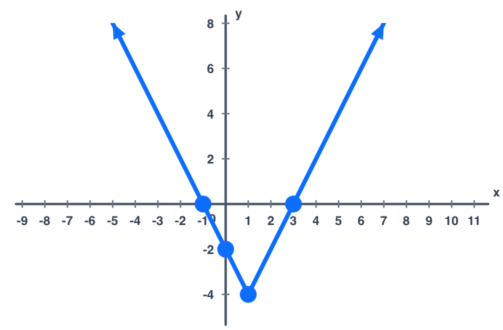
Example 5. For $f(x) = |x-2|$, solve $f(x) = 3$, $f(x) \le 3$, and $f(x) > 3$.

Answer: **Equality:** $x = -1, 5$. $f(x) \le 3$ **on** $[-1, 5]$. $f(x) > 3$ **on** $(-\infty, -1) \cup (5, \infty)$.

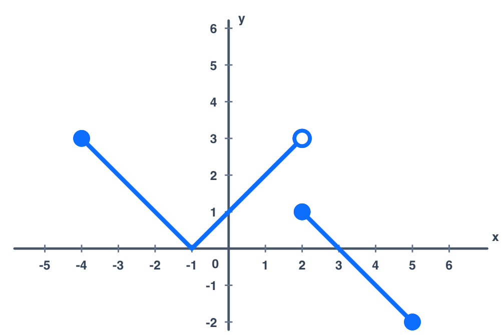


Example 6. Graph $p(x) = 2|x-1| - 4$, then state the vertex, intercepts, domain, and range.

Answer: **Vertex:** $(1, -4)$. **x-intercepts:** $(-1, 0)$, $(3, 0)$. **y-intercept:** $(0, -2)$. **Domain: all real numbers,** $(-\infty, \infty)$. **Range:** $[-4, \infty)$.



Example 7. Use the piecewise graph that includes an absolute value piece. State the domain and find $f(1)$.



Answer: **Domain:** $[-4, 5]$. $f(1) = 2$.

Example 8. Use the table for $a(x) = 2|x+1| - 5$. Identify the vertex and where the function is decreasing and increasing.

x	-4	-3	-2	-1	0	1	2
$a(x)$	1	-1	-3	-5	-3	-1	1

Answer: **The lowest output is -5 at $x = -1$, so the vertex is $(-1, -5)$. The function decreases on $(-\infty, -1)$ and increases on $(-1, \infty)$.**

HW 18 Radical Equations

Solve radical equations while tracking domain restrictions and extraneous solutions.

QUICK REFERENCE

- Even roots need the inside expression to be nonnegative when working with real numbers.
- Rational exponents and radicals are two ways to write the same idea: $a^{1/n} = \sqrt[n]{a}$ and $a^{m/n} = \sqrt[n]{a^m}$, whenever the real-number expression is defined.
- Use a consistent process when solving an equation with even roots.
  - Isolate the radical, raise both sides to the matching power, solve, and then check.
  - If a square root is isolated, the other side must be greater than or equal to 0. This can help reject impossible candidates.
  - Squaring can create extraneous solutions, which are answers that do not work in the original equation.
  - With radicals on both sides, you may need to square, simplify, isolate again, and square again.
- Odd roots, such as cube roots, can accept negative inputs and can be undone by cubing both sides.

Example 1. Find the real-number domain of $r(x) = \sqrt{2x-6}$.

Answer: **Domain: $[3, \infty)$.**

Example 2. Solve $\sqrt{3x+1} = 5$.

Answer: $x = 8$.

Example 3. Solve $\sqrt{x+4} + 2 = x$.

Answer: $x = 5$.

Example 4. Solve $\sqrt{x+5} = x - 1$ and check for extraneous solutions.

Answer: **Solution: $x = 4$. Extraneous candidate: $x = -1$.**

Example 5. Solve $\sqrt{x^2 - 4} = x + 1$ and check for extraneous solutions.

Answer: **No solution, $\varnothing$. The candidate $x = -\frac{5}{2}$ is extraneous.**

Example 6. Solve $\sqrt{x+9} = \sqrt{2x+1}$.

Answer: $x = 8$.

Example 7. Solve $\sqrt{x+1} + 1 = \sqrt{2x+4}$.

Answer: $x = 0$.

Example 8. Solve $\sqrt[3]{2x-1} = 3$.

Answer: $x = 14$.

Example 9. Rewrite $(3x+4)^{1/2} = 5$ using radical notation, then solve.

Answer: $\sqrt{3x+4} = 5$. **Solution:** $x = 7$.

Example 10. Use the check table for $\sqrt{x+5} = x - 1$. Which candidate is extraneous?

Candidate	$\sqrt{x+5}$	$x-1$	Works?
$x=-1$	2	-2	
$x=4$	3	3	

Answer: The extraneous candidate is $x = -1$, because it does not make the original equation true. The solution is $x = 4$.

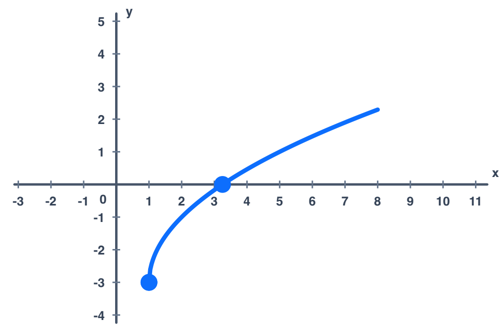
HW 19 Radical Functions and Graphs

Read square-root and cube-root function behavior from transformations and graph features.

QUICK REFERENCE

- Square-root and cube-root functions have different key points and domains.
  - A square-root graph starts at an endpoint and continues in one direction.
  - For $f(x) = a\sqrt{x-h} + k$, the starting point is (h, k), and the real-number domain begins at $x = h$.
  - A cube-root graph allows all real inputs and passes through a center point instead of starting at an endpoint.
  - For $f(x) = a\sqrt[3]{x-h} + k$, the center point is (h, k), and the domain is all real numbers, $(-\infty, \infty)$.
- This uses the same transformation language from parent functions: h shifts left or right, k shifts up or down, and a controls stretch, compression, or reflection. Replacing x with $-x$ reflects a graph across the y-axis.
- A radical in a denominator must be defined and cannot equal 0.
- When reading a radical graph, look for endpoint or center point, domain, range, intercepts, and increasing or decreasing behavior.
- Piecewise radical pieces may have restricted domains that must match the given interval.

Example 1. Use the graph of $s(x) = 2\sqrt{x-1} - 3$. State starting point, domain, range, and x-intercept.



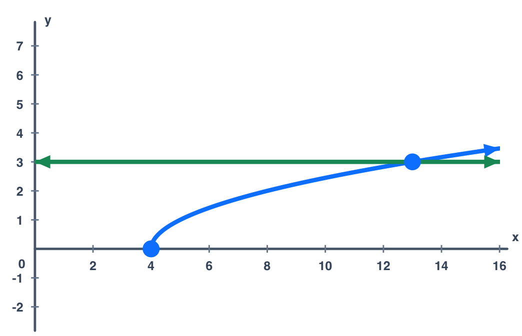
Answer: **Starting point:** $(1, -3)$. **Domain:** $[1, \infty)$. **Range:** $[-3, \infty)$. **x-intercept:** $\left(\frac{13}{4}, 0\right)$.

Example 2. For $c(x) = \sqrt[3]{x+8} - 2$, state center point, domain, and range.

Answer: **Center point:** $(-8, -2)$. **Domain: all real numbers,** $(-\infty, \infty)$. **Range: all real numbers,** $(-\infty, \infty)$.

Example 3. Solve graphically/algebraically: $\sqrt{x-4} = 3$.

Answer: $x = 13$.



Example 4. A radical piece is $f(x) = \sqrt{x+2}$ only for $-2 \leq x \leq 7$. State the piece domain and range.

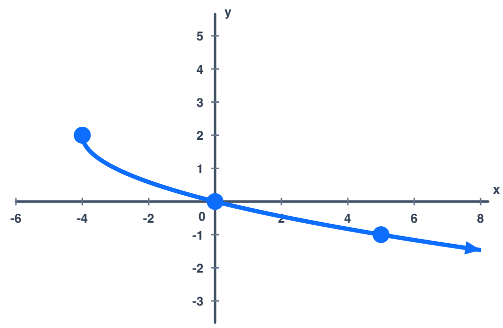
Answer: **Domain:** $[-2, 7]$. **Range:** $[0, 3]$.

Example 5. Find the domain of $h(x) = \dfrac{1}{\sqrt{x-5}}$.

Answer: **Domain:** $(5, \infty)$.

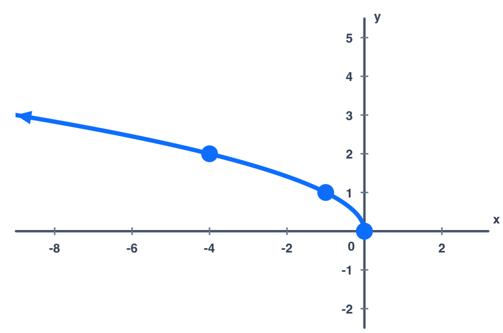
Example 6. Graph $r(x) = -\sqrt{x+4}+2$, then state starting point, domain, range, and whether the graph increases or decreases.

Answer: Starting point: $(-4, 2)$. **Domain:** $[-4, \infty)$. **Range:** $(-\infty, 2]$. **Decreasing.**



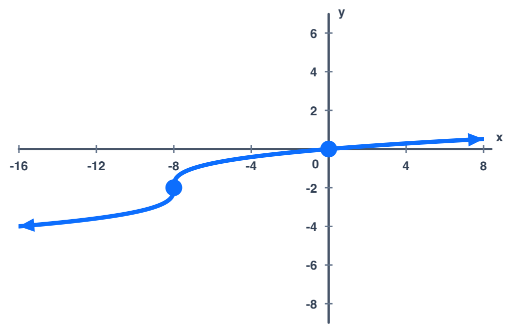
Example 7. Graph $q(x) = \sqrt{-x}$, then state the reflection, starting point, domain, range, and whether the graph increases or decreases.

Answer: Reflection: across the y-axis. Starting point: $(0, 0)$. **Domain:** $(-\infty, 0]$. **Range:** $[0, \infty)$. **Decreasing.**

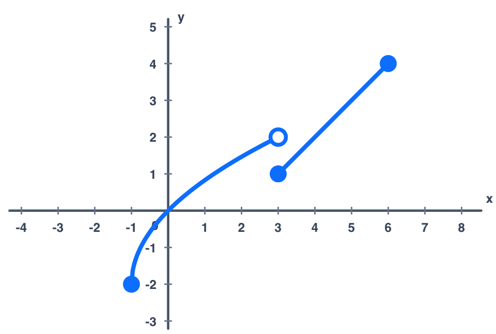


Example 8. Graph $c(x) = \sqrt[3]{x+8} - 2$, then state center point, domain, range, and intercepts.

Answer: **Center point:** $(-8, -2)$. **Domain: all real numbers,** $(-\infty, \infty)$. **Range: all real numbers,** $(-\infty, \infty)$. **x-intercept:** $(0, 0)$. **y-intercept:** $(0, 0)$.



Example 9. Use the piecewise graph with a radical piece. State the domain, the radical piece's range, and $f(3)$.



Answer: **Domain:** $[-1, 6]$. **Radical piece range:** $[-2, 2)$. $f(3) = 1$.

Example 10. Use the table for $r(x) = \sqrt{x+4} - 2$. State the starting point, domain, range, and x-intercept.

x	-4	-3	0	5
$r(x)$	-2	-1	0	1

Answer: **Starting point:** $(-4, -2)$. **Domain:** $[-4, \infty)$. **Range:** $[-2, \infty)$. **x-intercept:** $(0, 0)$.

Module 5: Polynomials

Students extend factoring, division, zeros, graphs, inequalities, and modeling to higher-degree polynomial functions.

HW 20 Polynomial Foundations and Factoring

Connect degree, leading coefficient, end behavior, and factoring patterns.

QUICK REFERENCE

- A polynomial is built from terms with nonnegative whole-number exponents.
- After simplifying and writing in descending powers, the leading term gives the degree and leading coefficient.
- Degree is the largest exponent after the polynomial is simplified.
- The leading coefficient and degree control end behavior.
  - Even degree ends go the same direction. Odd degree ends go opposite directions.
  - Even degree with a positive leading coefficient: as $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to \infty$.
  - Even degree with a negative leading coefficient: as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to -\infty$.
  - Odd degree with a positive leading coefficient: as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.
  - Odd degree with a negative leading coefficient: as $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to -\infty$.
- Use GCF, grouping, difference of squares, sum/difference of cubes, and trinomial patterns when factoring.
- Factoring completely may require more than one pattern.

Example 1. For $p(x) = -4x^5 + 7x^3 - 2x + 9$, identify leading term, degree, leading coefficient, and end behavior.

Answer: Leading term: $-4x^5$. Degree: 5. Leading coefficient: -4. End behavior: left end up, right end down. As $x \to -\infty$, $p(x) \to \infty$. As $x \to \infty$, $p(x) \to -\infty$.

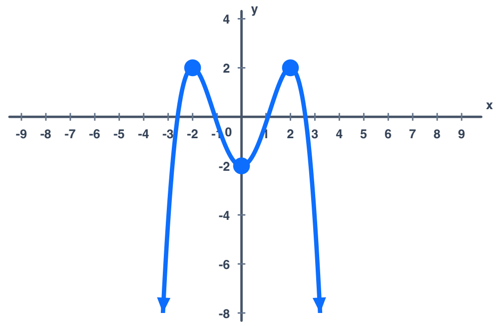
Example 2. Describe end behavior for $f(x) = 3x^6 - 2x + 1$ and $g(x) = -2x^4 + 5x - 8$.

Answer: f: both ends up; as $x \to \pm\infty$, $f(x) \to \infty$. g: both ends down; as $x \to \pm\infty$, $g(x) \to -\infty$.

Example 3. Describe end behavior for $h(x) = x^5 - 4x^2 + 1$ and $q(x) = -4x^5 + 7x^3 - 2x + 9$.

Answer: h: as $x \to -\infty$, $h(x) \to -\infty$; as $x \to \infty$, $h(x) \to \infty$. q: as $x \to -\infty$, $q(x) \to \infty$; as $x \to \infty$, $q(x) \to -\infty$.

Example 4. Use the graph to describe the end behavior and decide whether the polynomial has even or odd degree and positive or negative leading coefficient.



Answer: Both ends down. Even degree. Negative leading coefficient. As $x \to \pm\infty$, $f(x) \to -\infty$.

Example 5. Factor completely: $3x^4 - 12x^2$.

Answer: $3x^2(x-2)(x+2)$.

Example 6. Factor by grouping: $x^3 + 2x^2 - 9x - 18$.

Answer: $(x+2)(x-3)(x+3)$.

Example 7. Factor by grouping: $2x^3 + 6x^2 + 5x + 15$.

Answer: $(x+3)(2x^2+5)$.

Example 8. Factor $x^3 - 27$.

Answer: $(x-3)(x^2+3x+9)$.

Example 9. Factor completely: $2x^4 - 32$.

Answer: $2(x-2)(x+2)(x^2+4)$.

Example 10. Use the table to compare $x^3 - 4x$ with its factored form $x(x-2)(x+2)$. What zeros do both forms show?

x	-2	0	2	3
$x^3 - 4x$	0	0	0	15
$x(x-2)(x+2)$	0	0	0	15

Answer: Both forms show zeros at $x = -2$, $x = 0$, and $x = 2$.

HW 21 Polynomial Division

Use long division, synthetic division, the remainder theorem, and the factor theorem.

QUICK REFERENCE

- Polynomial division rewrites a polynomial as quotient plus remainder over divisor.
- Division can be checked with $f(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.
- Choose the division method that matches the divisor.
  - Long division works for any polynomial divisor.
  - Synthetic division is a shortcut for divisors of the form $x - c$.
- Use zero coefficients for missing powers so every place value is accounted for.
- Two theorems connect division, function values, and factors.
  - Remainder theorem: when dividing $f(x)$ by $x - c$, the remainder is $f(c)$.
  - Factor theorem: $x - c$ is a factor exactly when $f(c) = 0$.

Example 1. Divide $x^3 + 2x^2 - 5x + 6$ by $x + 3$ using both long and synthetic division.

Answer: **Quotient:** $x^2 - x - 2$. **Remainder:** 12.

Example 2. Use long division to divide $2x^4 + 3x^3 - x + 5$ by $x^2 + 1$.

Answer: **Quotient:** $2x^2 + 3x - 2$. **Remainder:** $-4x + 7$.

Example 3. Use synthetic division to divide $2x^3 - 3x^2 - 8x + 12$ by $x - 2$.

Answer: **Quotient:** $2x^2 + x - 6$. **Remainder:** 0.

Example 4. Use synthetic division to divide $3x^4 - 5x^3 + 2x - 8$ by $x - 2$.

Answer: **Quotient:** $3x^3 + x^2 + 2x + 6$. **Remainder:** 4.

Example 5. Use the remainder theorem to find the remainder when $f(x) = x^3 - 4x + 1$ is divided by $x + 2$.

Answer: **Remainder:** 1.

Example 6. Use the remainder theorem to find $f(-1)$ for $f(x) = 4x^5 - 2x^3 + 7x - 9$, and name the matching division remainder.

Answer: $f(-1) = -18$. **Remainder:** -18.

Example 7. Decide whether $x - 3$ is a factor of $f(x) = x^3 - 6x^2 + 11x - 6$.

Answer: **Yes,** $x - 3$ **is a factor.**

Example 8. Use the synthetic-division table for $2x^3 - 3x^2 - 8x + 12$ divided by $x - 2$. Identify the quotient and remainder.

Bring down / combine	2	-3	-8	12
Multiply by 2		4	2	-12
Bottom row	2			

Answer: Quotient: $2x^2 + x - 6$. **Remainder:** 0.

HW 22 Polynomial Zeros and Rational Zero Theorem

Use possible rational zeros, real/complex zeros, conjugate pairs, and factors.

QUICK REFERENCE

- The Rational Zero Theorem lists possible rational zeros as $\frac{p}{q}$, where p divides the constant term and q divides the leading coefficient.
- A zero c means $f(c) = 0$, and $x - c$ is a factor.
- After finding one zero, divide by its factor to lower the degree and keep solving.
- Polynomial zeros may be real or complex.
  - Real zeros may appear as x-intercepts on the graph.
  - For polynomials with real coefficients, nonreal complex zeros come in conjugate pairs.
- Writing factors from zeros reverses the zero-finding process. A zero c gives the factor $x - c$, and a zero with multiplicity m gives the repeated factor $(x - c)^m$.
- A degree n polynomial can have at most n real zeros.

Example 1. List possible rational zeros for $f(x) = 2x^3 - 3x^2 - 8x + 12$.

Answer: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}$.

Example 2. Find all zeros of $f(x) = x^3 - 6x^2 + 11x - 6$.

Answer: Zeros: $1, 2, 3$.

Example 3. Find all zeros of $f(x) = 2x^3 - 5x^2 - 4x + 3$.

Answer: Zeros: $-1, \frac{1}{2}, 3$.

Example 4. If 2 is a zero of $f(x) = x^3 - 3x^2 - 4x + 12$, find the remaining zeros.

Answer: Remaining zeros: 3 **and** -2.

Example 5. If $2+i$ is a zero of a polynomial with real coefficients, what other zero must appear?

Answer: $2-i$.

Example 6. The polynomial $f(x)=x^3-3x^2+4x-12$ has real coefficients and a zero of $2i$. Identify the conjugate zero, use the resulting quadratic factor to find the remaining zero, and write the polynomial in factored form.

Answer: **Conjugate zero:** $-2i$. **Quadratic factor:** x^2+4. **Remaining zero:** 3. **Factored form:** $(x-3)(x^2+4)$.

Example 7. Write a monic polynomial with zeros -1 of multiplicity 2, 4, and $3i$, assuming real coefficients.

Answer: $(x+1)^2(x-4)(x^2+9)$.

Example 8. A polynomial with real coefficients has zeros 5, $-2i$, and $2i$. Write factored form.

Answer: $(x-5)(x^2+4)$.

Example 9. Use the possible-zero table for $f(x)=2x^3-3x^2-8x+12$. Which tested values are zeros?

Candidate	-2	-1	1	2	3
$f(x)$	0	15	3	0	15

Answer: **Tested zeros:** $x=-2$ **and** $x=2$.

HW 23 Polynomial Graphs

Sketch and interpret polynomial graphs using end behavior, multiplicity, intercepts, and turning points.

QUICK REFERENCE

- Remember that end behavior comes from degree and leading coefficient.
- Zeros with odd multiplicity cross the x-axis. Zeros with even multiplicity touch and turn around at the x-axis.
- The y-intercept is $f(0)$.
- A degree n polynomial can have at most $n-1$ turning points.
- Polynomial domain is all real numbers, $(-\infty,\infty)$. Range depends on the graph.
- Factored form helps identify zeros and multiplicities quickly.
- A calculator graph and value table can help approximate zeros and turning points and verify increasing or decreasing intervals.

- Use the important features when creating or reading a polynomial graph.
  - A sketch should show end behavior, intercepts, and touch/cross behavior.
  - From a graph, be prepared to approximate turning points and determine increasing or decreasing intervals, domain, and range.

Example 1. For $f(x) = (x+2)^2(x-3)$, identify zeros and touch/cross behavior.

Answer: **Zero -2: touches. Zero 3: crosses.**

Example 2. For $h(x) = (x-2)^2(x+1)(x^2+4)$, identify the real zeros, multiplicities, x-intercepts, and touch/cross behavior. Explain what the factor x^2+4 contributes to the real graph.

Answer: **Real zero -1, multiplicity 1: x-intercept $(-1, 0)$, crosses. Real zero 2, multiplicity 2: x-intercept $(2, 0)$, touches. The factor x^2+4 has no real zeros, so it adds no x-intercepts.**

Example 3. For $g(x) = -(x-1)(x+4)^3$, state end behavior and zeros.

Answer: **End behavior: both ends down; as $x \to \pm\infty$, $g(x) \to -\infty$. Zeros: 1 and -4. Both cross.**

Example 4. Find the y-intercept of $p(x) = 2(x-1)^2(x+3)$.

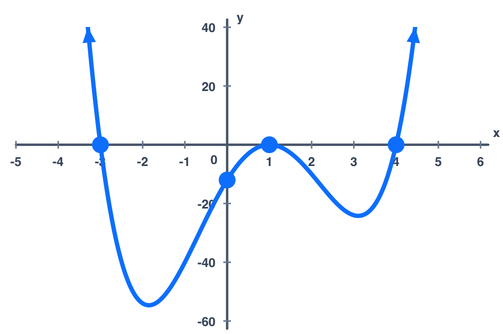
Answer: **y-intercept: $(0, 6)$.**

Example 5. A degree 5 polynomial has positive leading coefficient. Describe its end behavior and maximum possible turning points.

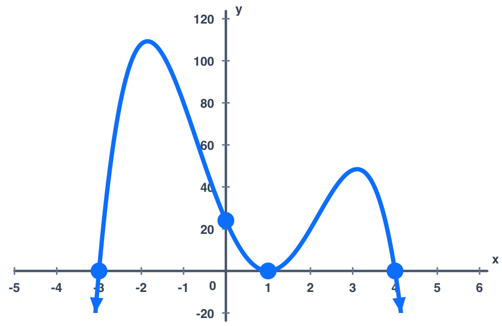
Answer: **End behavior: left end down, right end up. At most 4 turning points.**

Example 6. Sketch $f(x) = (x+3)(x-1)^2(x-4)$. Show end behavior, x-intercepts, touch/cross behavior, and y-intercept.

Answer: **End behavior: both ends up. x-intercepts: $(-3, 0)$ crosses, $(1, 0)$ touches, $(4, 0)$ crosses. y-intercept: $(0, -12)$.**



Example 7. Use the graph of $F(x) = -2(x+3)(x-1)^2(x-4)$. State degree, end behavior, x-intercepts with touch/cross behavior, and y-intercept.



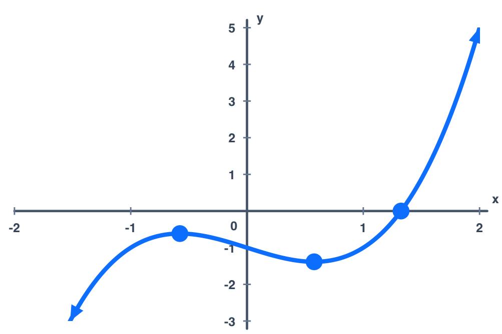
Answer: Degree: 4. End behavior: both ends down; as $x \to \pm\infty$, $F(x) \to -\infty$. x-intercepts: $(-3, 0)$ crosses, $(1, 0)$ touches, $(4, 0)$ crosses. y-intercept: $(0, 24)$.

Example 8. A polynomial graph has x-intercepts -3, 0, and 2, touches at $x = 0$, crosses at the others, and both ends go up. Give a possible factored form.

Answer: One possible function: $f(x) = x^2(x+3)(x-2)$.

Example 9. Use the graph and value table for $f(x) = x^3 - x - 1$ to approximate the real zero and turning points to the nearest hundredth. Then state where the function is increasing and decreasing.

x	-0.60	-0.58	0.58	1.32	1.33
$f(x)$	-0.616	-0.615	-1.385	-0.020	0.023



Answer: Real zero: $x \approx 1.32$. Local maximum: $(-0.58, -0.62)$. Local minimum: $(0.58, -1.38)$. Increasing on $(-\infty, -0.58) \cup (0.58, \infty)$. Decreasing on $(-0.58, 0.58)$.

Example 10. Use the sign table for $F(x) = -2(x+3)(x-1)^2(x-4)$. Where is the graph above the x-axis?

Interval	$(-\infty, -3)$	$(-3, 1)$	$(1, 4)$	$(4, \infty)$
Sign of $F(x)$	Negative	Positive	Positive	Negative

Answer: **Above the x-axis:** $(-3, 1) \cup (1, 4)$.

HW 24 Polynomial Inequalities

Use critical values, sign tables, interval notation, and positive/negative intervals.

QUICK REFERENCE

- Move all terms to one side so the polynomial is compared to 0.
- Use critical values to build and read a sign table.
  - Critical values are zeros of the polynomial, and they split the number line into sign intervals.
  - Use test points or multiplicity behavior to decide where the polynomial is positive or negative.
  - At an odd-multiplicity zero, the sign changes. At an even-multiplicity zero, the sign does not change.
  - Keep the intervals, test values, and signs organized in the table.
- Use brackets or closed points when equality is included. Use parentheses or open points when equality is not included.
- Some polynomial inequalities have all real numbers, $(-\infty, \infty)$, or no real solution, $\varnothing$.
- You can also determine inequality solutions from a polynomial graph by identifying where the graph is above or below the x-axis.

Example 1. Use the sign table to solve $x(x-2)(x+3) \leq 0$, then graph the solution on a number line.

Interval	$(-\infty, -3)$	$(-3, 0)$	$(0, 2)$	$(2, \infty)$
Test value	-4	-1	1	3
Sign	Negative	Positive	Negative	Positive

Answer: **Solution:** $(-\infty, -3] \cup [0, 2]$.

Example 2. Use a sign table to help solve $(x+2)(x-1)(x-4) > 0$.

Interval				
Test value				
Sign				

Answer: Solution: $(-2,1) \cup (4,\infty)$.

Interval	$(-\infty,-2)$	$(-2,1)$	$(1,4)$	$(4,\infty)$
Test value	-3	0	2	5
Sign	Negative	Positive	Negative	Positive

Example 3. Use a sign table to help solve $(x-3)^2(x+1) \leq 0$.

Interval			
Test value			
Sign			

Answer: Solution: $(-\infty,-1] \cup \{3\}$.

Interval	$(-\infty,-1)$	$(-1,3)$	$(3,\infty)$
Test value	-2	0	4
Sign	Negative	Positive	Positive

Example 4. Use a sign table to help solve $t^3 - 4t \geq 0$.

Interval				
Test value				
Sign				

Answer: Solution: $[-2,0] \cup [2,\infty)$.

Interval	$(-\infty,-2)$	$(-2,0)$	$(0,2)$	$(2,\infty)$
Test value	-3	-1	1	3
Sign	Negative	Positive	Negative	Positive

Example 5. Use a sign table to help find where $p(x) = (x+5)^2(x-2)$ is negative.

Interval			
Test value			
Sign			

Answer: $(-\infty, -5) \cup (-5, 2)$.

Interval	$(-\infty, -5)$	$(-5, 2)$	$(2, \infty)$
Test value	-6	0	3
Sign	Negative	Negative	Positive

Example 6. Use a sign table to help solve $(x+1)(x-2)^2(x-5) \leq 0$.

Interval				
Test value				
Sign				

Answer: Solution: $[-1, 5]$.

Interval	$(-\infty, -1)$	$(-1, 2)$	$(2, 5)$	$(5, \infty)$
Test value	-2	0	3	6
Sign	Positive	Negative	Negative	Positive

Example 7. Use a sign table to help solve $x^4 - 5x^2 + 4 > 0$.

Interval					
Test value					
Sign					

Answer: Solution: $(-\infty, -2) \cup (-1, 1) \cup (2, \infty)$.

Interval	$(-\infty, -2)$	$(-2, -1)$	$(-1, 1)$	$(1, 2)$	$(2, \infty)$
Test value	-3	$-\frac{3}{2}$	0	$\frac{3}{2}$	3
Sign	Positive	Negative	Positive	Negative	Positive

Example 8. Solve $x^4 + 1 > 0$ and $x^4 + 1 < 0$.

Answer: For $x^4 + 1 > 0$: all real numbers, $(-\infty, \infty)$. For $x^4 + 1 < 0$: no real solution, $\varnothing$.

HW 25 Polynomial Modeling and Applications

Use polynomial area, volume, and optimization-style interpretation.

QUICK REFERENCE

- Polynomial models often come from multiplying dimensions, such as length times width or length times width times height.
- Define the variable clearly so the expression matches the context.
- Volume and area models need realistic domain restrictions, even if the algebra allows more values.
- Zeros can represent when an area, volume, profit, or output becomes 0.
- Profit can be modeled by $P = R - C$, where R is revenue and C is cost.
- Maximum or minimum questions ask for the largest or smallest useful output in the context.
- When interpreting a maximum or minimum, include what the input and output mean.
- Polynomial regression or curve fitting gives an approximate model from data points.
- Compare a model output with observed data to see whether the model overestimates or underestimates the actual value.
- A model can be useful only on a realistic input interval, even if the graph continues forever.
- Threshold questions ask when the model is above or below a certain output value.
- Always check whether an answer is realistic in context.

Example 1. A rectangular garden has length $x + 5$ feet and width $x + 2$ feet, where x is positive. The garden needs an area of 70 square feet. Write and solve an equation for x.

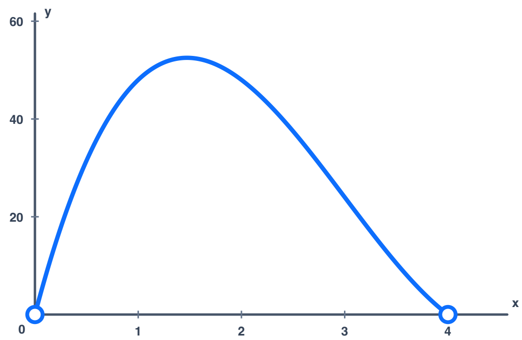
Answer: Equation: $(x + 5)(x + 2) = 70$. Expanded equation: $x^2 + 7x - 60 = 0$. Solution: $x = 5$ feet.

Example 2. A box has dimensions x, $x + 2$, and $10 - x$. Write a volume model and state a realistic domain.

Answer: $V(x) = x(x + 2)(10 - x)$. Realistic domain: $0 < x < 10$.

Example 3. A 10 in by 8 in sheet has squares of side x cut from each corner to make an open box. Write $V(x)$, give restrictions, and graph the model to estimate the maximum volume.

Answer: $V(x) = x(10-2x)(8-2x)$. **Restrictions:** $0 < x < 4$. **Maximum volume: about** 52.5 **cubic inches when** $x \approx 1.47$ **inches.**



Example 4. For $V(x) = x(8-2x)(10-2x)$, what values of x make the volume greater than 24 cubic inches? Interpret the result.

Answer: $V(x) > 24$ **for about** $0.35 < x < 3.00$. **The box volume is greater than** 24 **cubic inches when the cut size is between about** 0.35 **inch and** 3.00 **inches.**

Example 5. A business has revenue $R(x) = -x^3 + 15x^2 + 60x$ and cost $C(x) = 3x^2 + 96x$, where x is the number of items sold in hundreds. Write $P(x) = R(x) - C(x)$, factor, and find the break-even values.

Answer: $P(x) = -x^3 + 12x^2 - 36x = -x(x-6)^2$. **Break-even values:** $x = 0$ **and** $x = 6$, **meaning** 0 **items and** 600 **items.**

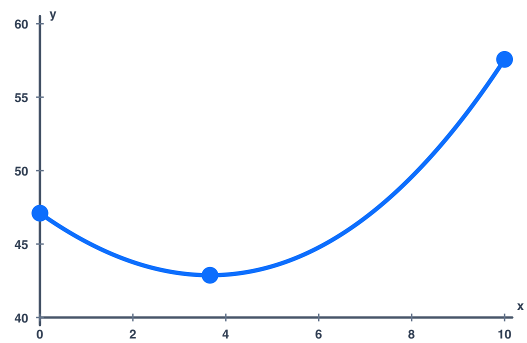
Example 6. A polynomial model $P(t) = 0.8t^3 - 20t^2 - 885t + 37815$ estimates population t years after 1970. Find and interpret $P(40)$. If the observed 2010 population was $21{,}900$, state whether the model overestimates or underestimates and by how much. Would this model be reasonable for predicting the population in 2200?

Answer: $P(40) = 21615$. **The model estimates** $21{,}615$ **people in 2010, which underestimates the observed population by** 285 **people. The model should not be used to predict 2200 because that is far outside the time period represented by the model.**

Example 7. The table shows outdoor temperature readings during one morning. Use cubic regression to find a model $T(x)$, where x is hours after midnight. Round coefficients to the nearest thousandth. Graph the model for $0 \leq x \leq 10$, estimate its turning point in that interval, and interpret the result.

Hours after midnight	0	2	4	6	8	10
Temperature $^\circ\mathrm{F}$	47	44	43	44	50	57

Answer: $T(x) \approx 0.005x^3 + 0.279x^2 - 2.243x + 47.103$. **The model has a minimum near** $(3.66, 42.88)$, **so the predicted low temperature is about** $42.9^\circ\mathrm{F}$ **around** 3.7 **hours after midnight.**



Module 6: Rational Expressions and Functions

Students simplify, solve, model, and graph rational relationships while tracking restrictions, holes, asymptotes, and sign intervals.

HW 26 Rational Expressions and Domains

Build rational-expression fluency while tracking restrictions from the start.

QUICK REFERENCE

- A rational expression is a fraction with polynomials.
- Find restrictions and factor before simplifying a rational expression.
  - Restrictions are denominator values that make the original denominator 0.
  - Restrictions come from the original expression, so a canceled factor can still give a restriction.
  - Factoring first makes common factors and restrictions visible.
  - Cancel common factors, not common terms.

- Use the operation to decide how to combine rational expressions.
  - Multiply by factoring and canceling common factors.
  - Divide by multiplying by the reciprocal, and remember the expression you divide by cannot equal 0.
  - Add and subtract by factoring denominators and using the least common denominator.

Example 1. State the restrictions for $\dfrac{x+4}{x^2-9}$.

Answer: **Restrictions:** $x \neq -3, 3$.

Example 2. State restrictions and simplify: $\dfrac{x^2-9}{x^2-x-6}$.

Answer: **Simplified form:** $\dfrac{x+3}{x+2}$. **Restrictions:** $x \neq 3, -2$.

Example 3. State restrictions and simplify: $\dfrac{3x^2+10x+3}{2x^2+7x+3}$.

Answer: **Simplified form:** $\dfrac{3x+1}{2x+1}$. **Restrictions:** $x \neq -3, -\dfrac{1}{2}$.

Example 4. Multiply and simplify: $\dfrac{n^2-4}{n^2+5n+6} \cdot \dfrac{n+3}{n-2}$.

Answer: **Simplified form:** 1. **Restrictions:** $n \neq -3, -2, 2$.

Example 5. Divide and simplify: $\dfrac{a^2-1}{a^2} \div \dfrac{a+1}{a}$.

Answer: **Simplified form:** $\dfrac{a-1}{a}$. **Restrictions:** $a \neq 0, -1$.

Example 6. Add and simplify: $\dfrac{2}{x} + \dfrac{3}{x+1}$.

Answer: **Simplified form:** $\dfrac{5x+2}{x(x+1)}$. **Restrictions:** $x \neq 0, -1$.

Example 7. Add and simplify: $\dfrac{1}{x-2} + \dfrac{3}{x^2-4}$.

Answer: **Simplified form:** $\dfrac{x+5}{(x-2)(x+2)}$. **Restrictions:** $x \neq 2, -2$.

Example 8. Subtract and simplify: $\dfrac{r}{r-4} - \dfrac{2}{r-4}$.

Answer: **Simplified form:** $\dfrac{r-2}{r-4}$. **Restrictions:** $r \neq 4$.

Example 9. State the domain of $\dfrac{x+5}{(x-2)(x+3)}$ in interval notation.

Answer: **Domain:** $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

HW 27 Rational Equations

Solve rational equations carefully and check for invalid or extraneous solutions.

QUICK REFERENCE

- A rational equation contains rational expressions and asks for values that make both sides equal.
- Use the restrictions and least common denominator throughout the solving process.
  - Factor the denominators first so the restrictions and least common denominator are clear.
  - List restrictions from every denominator.
  - Clear denominators by multiplying every term by the least common denominator.
  - Treat the results as candidate solutions and check them against the restrictions.
  - Reject any candidate that violates a restriction.
- The restriction check can lead to special solution sets.
  - Some rational equations have no solution, $\varnothing$, after restrictions are checked.
  - Some simplify to a statement that is true for every allowed input, giving all real numbers except the restrictions.

Example 1. Solve $\dfrac{3}{n} + \dfrac{1}{2} = \dfrac{5}{n}$.

Answer: $n = 4$.

Example 2. Solve $\dfrac{1}{x-2} = \dfrac{x}{x-2}$.

Answer: $x = 1$.

Example 3. Solve $\dfrac{x-5}{x+5} = \dfrac{x+1}{x+3}$.

Answer: $x = -\dfrac{5}{2}$.

Example 4. Solve $\dfrac{2}{x-3} + \dfrac{1}{x+3} = \dfrac{5x}{x^2-9}$.

Answer: $x = \dfrac{3}{2}$.

Example 5. Solve $\dfrac{3}{x+2} + \dfrac{2}{x+3} = \dfrac{5}{x^2+5x+6}$.

Answer: $x = -\dfrac{8}{5}$.

Example 6. Solve $\dfrac{1}{2x+1} + \dfrac{3}{x+3} = \dfrac{4}{2x^2+7x+3}$.

Answer: $x = -\dfrac{2}{7}$.

Example 7. Solve $\dfrac{c+1}{3c+18}+\dfrac{c}{2c+12}=\dfrac{3}{4c+24}$.

Answer: $c=\dfrac{1}{2}$.

Example 8. Solve $\dfrac{1}{x-2}-\dfrac{x+2}{x^2+2x+4}=\dfrac{14}{x^3-8}$.

Answer: $x=3$.

Example 9. Solve $\dfrac{2}{x+1}+\dfrac{3}{x-1}=1$.

Answer: $x=\dfrac{5\pm\sqrt{33}}{2}$.

Example 10. Solve $\dfrac{w}{w-4}=\dfrac{4}{w-4}+2$.

Answer: The candidate $w=4$ is excluded. No solution, $\varnothing$.

Example 11. Solve $\dfrac{1}{x-2}=\dfrac{1}{x-2}+3$.

Answer: No solution, $\varnothing$.

Example 12. Solve $\dfrac{x+1}{x-2}=1+\dfrac{3}{x-2}$.

Answer: All real numbers except $x=2$, $(-\infty,2)\cup(2,\infty)$.

HW 28 Rational Functions and Graphs

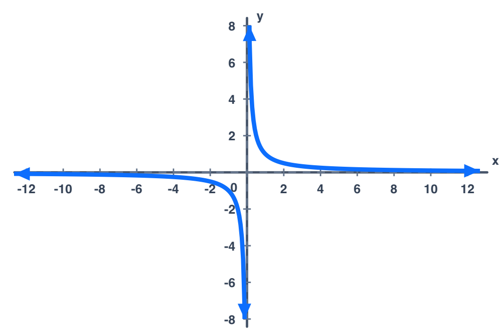
Build and read rational graphs using transformations, holes, asymptotes, intercepts, domain, range, and end behavior.

QUICK REFERENCE

- A rational function is a quotient of polynomials, and its denominator cannot equal 0.
- The parent function $f(x)=\dfrac{1}{x}$ has vertical asymptote $x=0$ and horizontal asymptote $y=0$.
- For $f(x)=\dfrac{a}{x-h}+k$, the vertical asymptote is $x=h$ and the horizontal asymptote is $y=k$. The value of a controls reflection and stretch or compression.
- The domain excludes values that make the original denominator 0.
- Simplify before deciding whether a denominator zero creates a hole or a vertical asymptote.
  - A canceled denominator factor creates a hole.
  - A denominator factor that remains creates a vertical asymptote.
  - If every denominator factor cancels, the graph may be a polynomial with a hole and no vertical, horizontal, or slant asymptote.

- x-intercepts come from numerator zeros that remain after simplifying. The y-intercept is $f(0)$, if 0 is in the domain.
- Compare numerator and denominator degrees to find a horizontal asymptote.
  - A smaller numerator degree gives $y = 0$.
  - Equal degrees use the ratio of the leading coefficients.
  - A larger numerator degree gives no horizontal asymptote.
- A slant asymptote can occur when the numerator degree is exactly one more than the denominator degree.
- Vertical asymptotes describe one-sided behavior. Horizontal and slant asymptotes describe end behavior as $x \to \infty$ and $x \to -\infty$.
- Use the important features to sketch a rational graph.
  - Factor and simplify first.
  - Mark holes and asymptotes.
  - Find the intercepts.
  - Use a point in each section of the graph.
- Read range from the completed graph. A horizontal asymptote or the y-value of a hole may be excluded, but check whether the graph reaches that output somewhere else.
- A table or calculator graph can help verify behavior near asymptotes and between critical x-values.

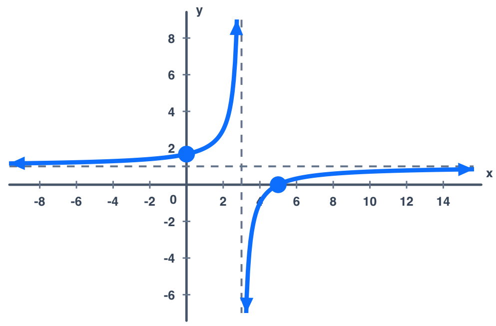
Example 1. Use the graph of the parent function $f(x) = \dfrac{1}{x}$. State its domain, range, vertical asymptote, horizontal asymptote, and the sign of each branch.



Answer: Domain: $(-\infty, 0) \cup (0, \infty)$. Range: $(-\infty, 0) \cup (0, \infty)$. Vertical asymptote: $x = 0$. Horizontal asymptote: $y = 0$. The left branch is negative and the right branch is positive.

Example 2. Graph $f(x) = -\dfrac{2}{x-3} + 1$. Then state its transformations, domain, range, asymptotes, and intercepts.

Answer: **Reflect across the x-axis, stretch by 2, shift right 3, and shift up 1. Domain: $(-\infty, 3) \cup (3, \infty)$. Range: $(-\infty, 1) \cup (1, \infty)$. Asymptotes: $x = 3$ and $y = 1$. Intercepts: $(5, 0)$ and $\left(0, \dfrac{5}{3}\right)$.**



Example 3. State the end-behavior asymptote (horizontal or slant) for each function: $a(x) = \dfrac{2}{x^2+1}$, $b(x) = \dfrac{3x^2-1}{x^2+4}$, and $c(x) = \dfrac{x^2+1}{x-2}$.

Answer: **$a(x)$: horizontal asymptote $y = 0$. $b(x)$: horizontal asymptote $y = 3$. $c(x)$: no horizontal asymptote; slant asymptote $y = x + 2$.**

Example 4. For $q(x) = \dfrac{x^2+3x+2}{x^2-1}$, identify the restrictions, hole, vertical asymptote, and horizontal asymptote.

Answer: **Restrictions: $x \neq -1, 1$. Hole: $\left(-1, -\dfrac{1}{2}\right)$. Vertical asymptote: $x = 1$. Horizontal asymptote: $y = 1$.**

Example 5. For $p(x) = \dfrac{x^3-64}{x^2-5x+4}$, identify the restrictions, hole, vertical asymptote, and slant asymptote.

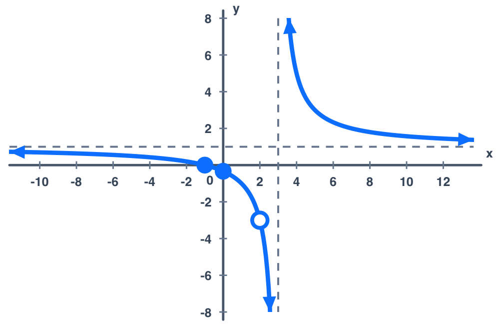
Answer: **Restrictions: $x \neq 1, 4$. Hole: $(4, 16)$. Vertical asymptote: $x = 1$. Slant asymptote: $y = x + 5$.**

Example 6. Describe the graph of $m(x) = \dfrac{12x^2+7x-10}{4x+5}$. State its domain, hole, and asymptotes.

Answer: **The graph is the line $y = 3x - 2$ with a hole at $\left(-\dfrac{5}{4}, -\dfrac{23}{4}\right)$. Domain: $(-\infty, -\dfrac{5}{4}) \cup (-\dfrac{5}{4}, \infty)$. It has no vertical, horizontal, or slant asymptote.**

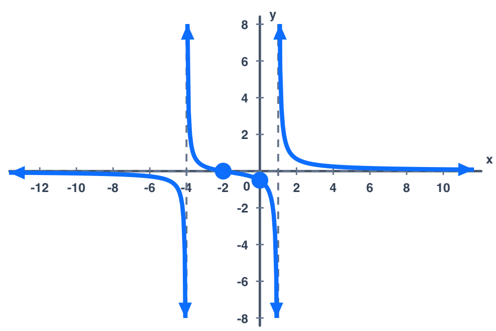
Example 7. Graph $r(x) = \dfrac{(x-2)(x+1)}{(x-2)(x-3)}$. Then state its domain, range, hole, asymptotes, and intercepts.

Answer: **Domain:** $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$. **Range:** $(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$. **Hole:** $(2, -3)$. **Asymptotes:** $x = 3$ and $y = 1$. **Intercepts:** $(-1, 0)$ and $\left(0, -\dfrac{1}{3}\right)$.

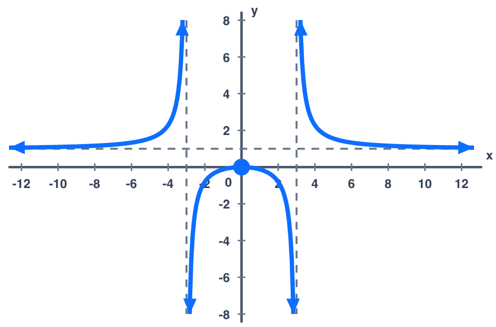


Example 8. Graph $h(x) = \dfrac{x+2}{(x-1)(x+4)}$. Then state its domain, range, vertical asymptotes, horizontal asymptote, and intercepts.

Answer: **Domain:** $(-\infty, -4) \cup (-4, 1) \cup (1, \infty)$. **Range: all real numbers,** $(-\infty, \infty)$. **Vertical asymptotes:** $x = -4$ and $x = 1$. **Horizontal asymptote:** $y = 0$. **Intercepts:** $(-2, 0)$ and $\left(0, -\dfrac{1}{2}\right)$.



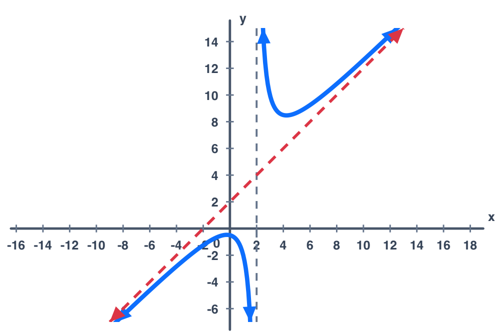
Example 9. Use the graph of $u(x) = \dfrac{x^2}{x^2 - 9}$. State its domain, range, vertical asymptotes, horizontal asymptote, and intercepts.



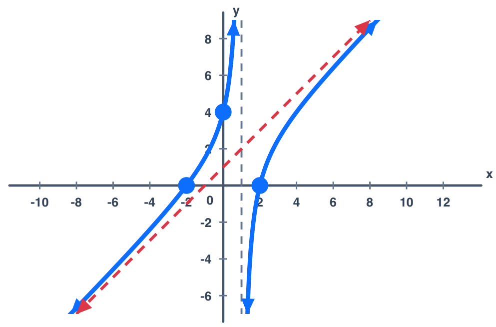
Answer: Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. Range: $(-\infty, 0] \cup (1, \infty)$. Vertical asymptotes: $x = -3$ and $x = 3$. Horizontal asymptote: $y = 1$. The x- and y-intercept are both $(0, 0)$.

Example 10. Graph $s(x) = \dfrac{x^2 + 1}{x - 2}$, including its vertical and slant asymptotes.

Answer: Vertical asymptote: $x = 2$. Slant asymptote: $y = x + 2$. Domain: $(-\infty, 2) \cup (2, \infty)$. No real x-intercepts. y-intercept: $\left(0, -\dfrac{1}{2}\right)$.



Example 11. Use the graph of $k(x) = \dfrac{x^2 - 4}{x - 1}$. State its domain, vertical asymptote, slant asymptote, intercepts, and end behavior.



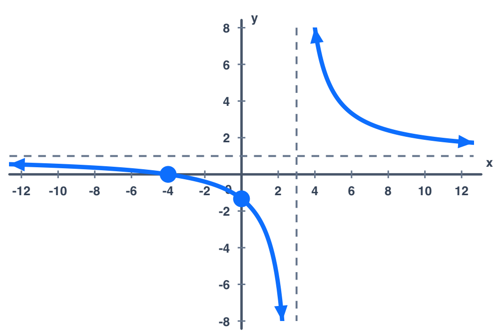
Answer: Domain: $(-\infty, 1) \cup (1, \infty)$. Vertical asymptote: $x = 1$. Slant asymptote: $y = x + 1$. Intercepts: $(-2, 0)$, $(2, 0)$, and $(0, 4)$. As $x \to \infty$ and $x \to -\infty$, the graph approaches $y = x + 1$.

Example 12. Use the table for $g(x) = \dfrac{x + 4}{x - 3}$ to describe the one-sided behavior near $x = 3$. Write the behavior using limit notation.

x	2.5	2.9	2.99	3.01	3.1	3.5
$g(x)$	-13	-69	-699	701	71	15

Answer: As $x \to 3^-$, $g(x) \to -\infty$. As $x \to 3^+$, $g(x) \to \infty$.

Example 13. Use the graph of $g(x) = \dfrac{x + 4}{x - 3}$. State its domain, range, asymptotes, intercepts, and end behavior.



Answer: Domain: $(-\infty, 3) \cup (3, \infty)$. Range: $(-\infty, 1) \cup (1, \infty)$. Asymptotes: $x = 3$ and $y = 1$. Intercepts: $(-4, 0)$ and $\left(0, -\dfrac{4}{3}\right)$. As $x \to \pm\infty$, $g(x) \to 1$.

HW 29 Variation and Rational Function Applications

Write variation models and interpret rational models through values, graphs, asymptotes, and extrema.

QUICK REFERENCE

- Translate the variation statement into an equation before substituting numbers.
- Use the wording to identify the type of variation.
  - Direct variation has the form $y = kx$. The ratio $\frac{y}{x}$ stays constant.
  - Inverse variation has the form $y = \frac{k}{x}$. The product xy stays constant.
  - Joint variation means one variable varies directly with the product of two or more variables.
  - Combined variation mixes direct and inverse variation in one model.
- Use the given values to find the constant of variation k, then use the completed model for the new situation.
- For a rational model, use the context to choose a realistic domain. Time, measurements, and numbers of items are usually nonnegative or positive.
- Tables and graphs show how a rational model changes over the realistic domain.
- A horizontal asymptote describes the value a model approaches over time or as the input grows. State what that value means in context.
- A graph can be used to estimate a maximum or minimum when an exact algebraic method is not required.
- Interpret final values using the quantities and units from the problem.

Example 1. Distance d varies directly with time t. A vehicle travels 150 miles in 3 hours at a constant speed. Find the model and the distance traveled in 7 hours.

Answer: Model: $d = 50t$. In 7 hours, the vehicle travels 350 miles.

Example 2. For a fixed 240-mile trip, travel time t varies inversely with speed r. Find the model and the travel time at 80 mph.

Answer: Model: $t = \frac{240}{r}$. At 80 mph, the travel time is 3 hours.

Example 3. The volume V of a rectangular box varies jointly with its length l, width w, and height h. If $V = 120$ when $l = 4$, $w = 5$, and $h = 6$, find V when $l = 8$, $w = 3$, and $h = 7$.

Answer: Model: $V = lwh$. The new volume is 168 cubic units.

Example 4. Pressure P varies directly with temperature T and inversely with volume V. If $P = 12$ when $T = 300$ and $V = 25$, find P when $T = 350$ and $V = 20$.

Answer: Model: $P = \frac{T}{V}$. The new pressure is 17.5 units.

Example 5. Use the tables to decide which relationship is direct variation and which is inverse variation. Then write the variation equation for each model.

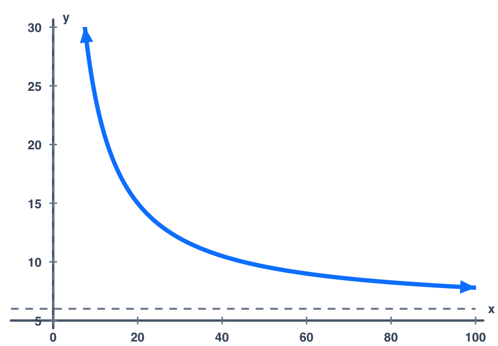
Model	$x = 2$	$x = 4$	$x = 8$	Pattern
A: y	6	12	24	$\frac{y}{x} = 3$
B: y	24	12	6	$xy = 48$

Answer: **Model A is direct variation: $y = 3x$. Model B is inverse variation: $y = \frac{48}{x}$.**

Example 6. The average cost per reusable bottle is $A(n) = \frac{6n + 180}{n}$, where $n > 0$. Find $A(10)$, $A(30)$, and $A(90)$, graph the model, state the horizontal asymptote, and interpret it.

Answer: **Horizontal asymptote: $A(n) = 6$. As production increases, the average cost approaches $\$6$ per bottle.**

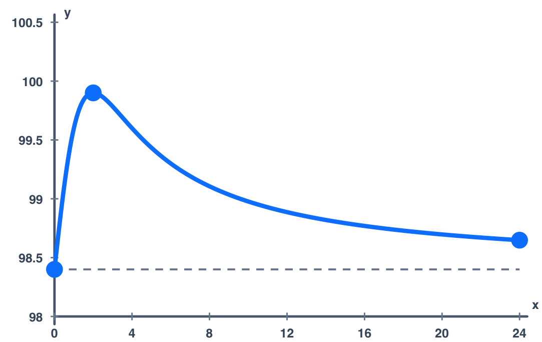
Bottles n	10	30	90
Average cost $A(n)$	$\$24$	$\$12$	$\$8$



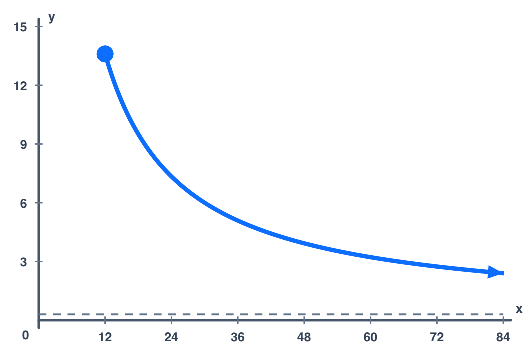
Example 7. During an illness, a patient's temperature is modeled by $T(t) = \dfrac{6t}{t^2+4} + 98.4$, where t is hours after symptoms begin. Find $T(t)$ at $t = 0, 1, 2, 4, 8,$ and 24, graph the model on $[0, 24]$, find the maximum temperature, and interpret the horizontal asymptote. Round temperatures to the nearest tenth.

Answer: **Maximum:** $99.9°\mathrm{F}$ **at** $t = 2$ **hours. Horizontal asymptote:** $T = 98.4$. **Over time, the model approaches a temperature of** $98.4°\mathrm{F}$.

Time t	0	1	2	4	8	24
Temperature $T(t)$	$98.4°\mathrm{F}$	$99.6°\mathrm{F}$	$99.9°\mathrm{F}$	$99.6°\mathrm{F}$	$99.1°\mathrm{F}$	$98.6°\mathrm{F}$



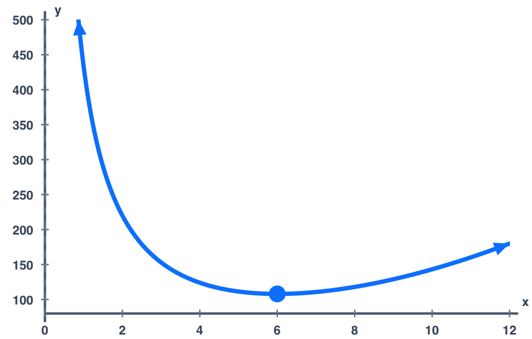
Example 8. Use the graph of the medication model $M(t) = \dfrac{1.2t + 720}{4t + 6}$, $t \geq 12$, where $M(t)$ is concentration in parts per million and t is time in hours. Find $M(12)$, state the horizontal asymptote, and interpret it.



Answer: $M(12) = 13.6$ **parts per million. Horizontal asymptote:** $M = 0.3$. **As time continues, the concentration approaches** 0.3 **parts per million.**

Example 9. An open-top box with a square base must hold 108 cubic centimeters. Let x be the side length of the base. Write surface area S as a function of x, graph the model for $x > 0$, and find the dimensions that minimize surface area.

Answer: $S(x) = x^2 + \dfrac{432}{x}$, with $x > 0$. Minimum surface area: $108\ \mathrm{cm}^2$ when $x = 6\ \mathrm{cm}$. The height is $3\ \mathrm{cm}$, so the box dimensions are $6\ \mathrm{cm} \times 6\ \mathrm{cm} \times 3\ \mathrm{cm}$.



HW 30 Rational Inequalities

Use critical values, excluded values, sign tables, and interval notation.

QUICK REFERENCE

- A rational inequality compares a rational expression to 0, or can be rewritten that way.
- Use all numerator and denominator zeros to build the sign table.
  - If the inequality is not already compared to 0, move all terms to one side first.
  - Critical values include numerator zeros and denominator zeros.
  - Use test points on the intervals made by all critical values.
  - A factor with even multiplicity does not change sign at its zero. A factor with odd multiplicity does change sign.
  - Include numerator zeros only when equality is allowed. Denominator zeros are excluded values and are never included.
- A graph can show where the rational function is above, below, or on the x-axis.

Example 1. Solve $\dfrac{x-3}{x+2} \geq 0$.

Answer: Critical values: -2 excluded, 3 zero. Solution: $(-\infty, -2) \cup [3, \infty)$.

Example 2. Solve $\dfrac{x+1}{(x-4)(x+2)} < 0$.

Answer: Critical values: $-2, -1, 4$. Solution: $(-\infty, -2) \cup (-1, 4)$.

Example 3. Solve $\dfrac{x^2-9}{x-1} \leq 0.$

Answer: Critical values: $-3, 1, 3$, with 1 excluded. Solution: $(-\infty, -3] \cup (1, 3]$.

Example 4. Explain why $x = 4$ cannot be included in a solution to $\dfrac{x+2}{x-4} \leq 0.$

Answer: At $x = 4$, the denominator is 0, so the expression is undefined.

Example 5. Solve $\dfrac{x-2}{x+1} > 3.$

Answer: Equivalent inequality: $\dfrac{-2x-5}{x+1} > 0$. Critical values: $-\dfrac{5}{2}$ and -1. Solution: $\left(-\dfrac{5}{2}, -1\right)$.

Example 6. Solve $\dfrac{1}{x-2} + \dfrac{2}{x+1} \leq 0.$

Answer: Equivalent inequality: $\dfrac{3(x-1)}{(x-2)(x+1)} \leq 0$. Critical values: $-1, 1, 2$, with -1 and 2 excluded. Solution: $(-\infty, -1) \cup [1, 2)$.

Example 7. Use the blank sign table to solve $\dfrac{(x-1)(x+3)}{(x-4)(x+2)} \geq 0.$

Interval					
Test value					
Sign					

Answer: Critical values: $-3, -2, 1, 4$, with -2 and 4 excluded. Solution: $(-\infty, -3] \cup (-2, 1] \cup (4, \infty)$.

Interval	$(-\infty, -3)$	$(-3, -2)$	$(-2, 1)$	$(1, 4)$	$(4, \infty)$
Test value	-4	$-\frac{5}{2}$	0	2	5
Sign	Positive	Negative	Positive	Negative	Positive

Example 8. A rational graph crosses the x-axis at $x = -1$ and $x = 5$, has vertical asymptotes at $x = 2$ and $x = 7$, and is below the x-axis on $(-1, 2)$ and $(5, 7)$. Solve $f(x) < 0$.

Answer: Use the intervals where the graph is below the x-axis: $(-1, 2) \cup (5, 7)$. Do not include 2 or 7 because they are asymptotes, and do not include -1 or 5 because the inequality is strict.

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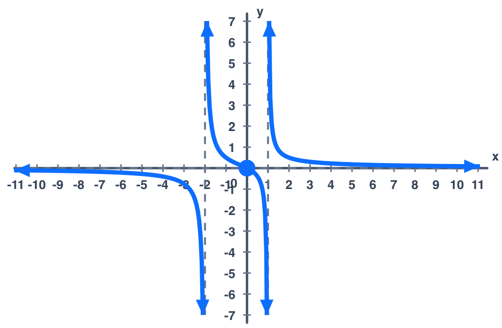
Example 9. Use the graph of $f(x) = \dfrac{x}{(x+2)(x-1)}$ to solve $f(x) > 0$ and $f(x) \leq 0$.



Answer: $f(x) > 0$ **on** $(-2, 0) \cup (1, \infty)$. $f(x) \leq 0$ **on** $(-\infty, -2) \cup [0, 1)$.

Example 10. Use the blank sign table to solve $\dfrac{(x-6)^2}{x^2 - 25} \geq 0$.

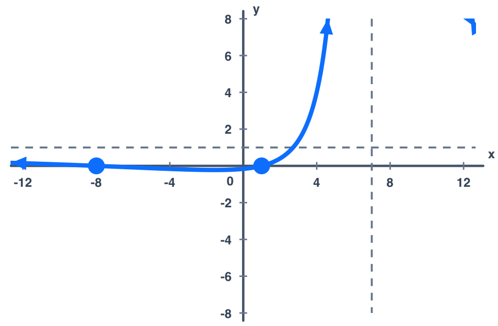
Interval				
Test value				
Sign				

Answer: Critical values: $-5, 5, 6$**, with** -5 **and** 5 **excluded. The sign does not change at the repeated zero** $x = 6$**. Solution:** $(-\infty, -5) \cup (5, \infty)$.

Interval	$(-\infty, -5)$	$(-5, 5)$	$(5, 6)$	$(6, \infty)$
Test value	-6	0	$\frac{11}{2}$	7
Sign	Positive	Negative	Positive	Positive

Example 11. Graph $g(x) = \dfrac{x^2+7x-8}{x^2-14x+49}$, then use the graph to solve $g(x) \geq 0$.

Answer: Zeros: $x=-8$ **and** $x=1$**. Excluded value:** $x=7$**. Solution:** $(-\infty,-8] \cup [1,7) \cup (7,\infty)$.



Example 12. Use the sign table to solve $\dfrac{x-1}{x+3} \leq 0$.

Interval			
Test value			
Sign			

Answer: The expression is negative between -3 **and** 1**, and equals** 0 **at** $x=1$**. Exclude** $x=-3$**. Solution:** $(-3,1]$.

Interval	$(-\infty,-3)$	$(-3,1)$	$(1,\infty)$
Test value	-4	0	2
Sign	Positive	Negative	Positive

Module 7: Function Operations, Composition, and Inverses

Students combine functions, use the difference quotient, and reverse functions through formulas, tables, and graphs.

HW 31 Function Operations and Composition

Combine functions, track domains, and compose functions in the correct order.

QUICK REFERENCE

- Function operations combine outputs at the same input.
  - Sum: $(f+g)(x)=f(x)+g(x)$.
  - Difference: $(f-g)(x)=f(x)-g(x)$.
  - Product: $(fg)(x)=f(x)g(x)$.
  - Quotient: $\left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}$.
  - For a sum, difference, or product, use inputs allowed by both functions. For a quotient, also exclude inputs that make the denominator function equal 0.
- Composition means placing one function inside another, not multiplying: $(f\circ g)(x)=f(g(x))$.
  - Work from the inside out.
  - Replace every x in the outside function with the entire inside expression. Parentheses help keep the substitution together.
  - Order matters. Usually $f(g(x))\neq g(f(x))$.
  - The input must work in the inside function, and the inside function's output must work in the outside function.
  - A simplified composition can hide a restriction. Keep any restrictions from the original functions.

Example 1. Let $f(x)=x^2-4$ and $g(x)=x-2$. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, and $\left(\dfrac{f}{g}\right)(x)$. State the domain of each.

Answer: $(f+g)(x)=x^2+x-6$, **domain: all real numbers,** $(-\infty,\infty)$. $(f-g)(x)=x^2-x-2$, **domain: all real numbers,** $(-\infty,\infty)$. $(fg)(x)=x^3-2x^2-4x+8$, **domain: all real numbers,** $(-\infty,\infty)$. $\left(\dfrac{f}{g}\right)(x)=x+2$, **domain:** $(-\infty,2)\cup(2,\infty)$.

Example 2. Let $f(x)=\sqrt{x+2}$ and $g(x)=x-3$. Find $(f+g)(x)$ and $\left(\dfrac{f}{g}\right)(x)$, including the domain of each.

Answer: $(f+g)(x)=\sqrt{x+2}+x-3$, **domain:** $[-2,\infty)$. $\left(\dfrac{f}{g}\right)(x)=\dfrac{\sqrt{x+2}}{x-3}$, **domain:** $[-2,3)\cup(3,\infty)$.

Example 3. Let $f(x)=3x-1$ and $g(x)=x^2+2$. Evaluate $(f\circ g)(2)$, $(g\circ f)(2)$, $(f\circ f)(2)$, and $(g\circ g)(2)$.

Answer: $(f\circ g)(2)=17$. $(g\circ f)(2)=27$. $(f\circ f)(2)=14$. $(g\circ g)(2)=38$.

Example 4. Let $f(x) = 2x + 3$ and $g(x) = x^2 - 1$. Find $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$, including the domain of each.

Answer: $(f \circ g)(x) = 2x^2 + 1$. $(g \circ f)(x) = 4x^2 + 12x + 8$. $(f \circ f)(x) = 4x + 9$. $(g \circ g)(x) = x^4 - 2x^2$. Each domain is all real numbers, $(-\infty, \infty)$.

Example 5. Use the tables for f and g to find $(f + g)(2)$, $(fg)(2)$, and $(f \circ g)(2)$.

x	0	1	2	3
$f(x)$	5	4	1	0
$g(x)$	2	3	1	0

Answer: $(f + g)(2) = 2$. $(fg)(2) = 1$. $(f \circ g)(2) = 4$.

Example 6. A store takes 20% off an item and charges a flat $\$5$ shipping fee. Let $d(x) = 0.80x$ and $s(x) = x + 5$. Find $(s \circ d)(50)$ and $(d \circ s)(50)$. Which composition represents applying the discount first and then adding shipping?

Answer: $(s \circ d)(50) = \$45$. $(d \circ s)(50) = \$44$. Applying the discount first and then adding shipping is $(s \circ d)(50)$.

Example 7. Let $f(x) = \sqrt{x}$ and $g(x) = x - 5$. Find $(f \circ g)(x)$ and its domain.

Answer: $(f \circ g)(x) = \sqrt{x - 5}$. Domain: $[5, \infty)$.

Example 8. Using the same functions, find $(g \circ f)(x)$ and its domain.

Answer: $(g \circ f)(x) = \sqrt{x} - 5$. Domain: $[0, \infty)$.

Example 9. Let $f(x) = \dfrac{1}{x}$ and $g(x) = x - 3$. Find $(f \circ g)(x)$ and its domain.

Answer: $(f \circ g)(x) = \dfrac{1}{x - 3}$, with $x \neq 3$.

Example 10. Let $f(x) = \sqrt{x}$ and $g(x) = x^2 - 9$. Find $(f \circ g)(x)$, $(g \circ f)(x)$, and both domains.

Answer: $(f \circ g)(x) = \sqrt{x^2 - 9}$, domain: $(-\infty, -3] \cup [3, \infty)$. $(g \circ f)(x) = x - 9$, domain: $[0, \infty)$.

Example 11. Let $f(x) = \dfrac{6}{x - 7}$ and $g(x) = \dfrac{1}{x}$. Find $(f \circ g)(x)$, $(g \circ f)(x)$, and $(g \circ g)(x)$, including the domain of each.

Answer: $(f \circ g)(x) = \dfrac{6x}{1 - 7x}$, domain: $(-\infty, 0) \cup \left(0, \dfrac{1}{7}\right) \cup \left(\dfrac{1}{7}, \infty\right)$. $(g \circ f)(x) = \dfrac{x - 7}{6}$, domain: $(-\infty, 7) \cup (7, \infty)$. $(g \circ g)(x) = x$, domain: $(-\infty, 0) \cup (0, \infty)$.

HW 32 Difference Quotient

Use the difference quotient with simple linear and quadratic functions.

QUICK REFERENCE

- The difference quotient is $\dfrac{f(x+h)-f(x)}{h}$.
- It gives the average rate of change from x to $x+h$. In Calculus I, this idea leads to derivatives.
- Keep the substitution and simplification organized.
  - Substitute $x+h$ everywhere the function has x, and use parentheses to keep the substitution together.
  - Keep $f(x+h)$ and $f(x)$ grouped carefully. The subtraction in front of $f(x)$ changes the sign of every term in that expression.
  - Simplify the numerator, factor out h, and cancel the common factor.
  - Use $h \neq 0$, because the denominator cannot be 0.

Example 1. Find and simplify the difference quotient for $f(x)=4x-7$.

Answer: 4, with $h \neq 0$.

Example 2. Find and simplify the difference quotient for $f(x)=2x^2-5$.

Answer: $4x+2h$, with $h \neq 0$.

Example 3. Find and simplify the difference quotient for $f(x)=x^2+3x+7$.

Answer: $2x+h+3$, with $h \neq 0$.

HW 33 Inverse Functions

Check one-to-one behavior, find inverse formulas, and connect inverse domains and ranges.

QUICK REFERENCE

- An inverse function reverses the input and output of the original function.
- The notation $f^{-1}(x)$ means the inverse function. It does not mean $\dfrac{1}{f(x)}$.
- A function has an inverse function only if it is one-to-one, meaning each output comes from only one input.
  - The horizontal line test checks one-to-one behavior on a graph.
  - A quadratic needs a restricted domain before it can have an inverse function. Choose the square-root branch that matches that restriction.

- Find and check an inverse using its algebraic and graphical connections.
  - To find an inverse formula, replace $f(x)$ with y, swap x and y, then solve for y.
  - To verify an inverse, both compositions should return the original input: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
  - The domain of a function becomes the range of its inverse, and the range becomes the domain.
  - Inverse graphs reflect across the line $y = x$.

Example 1. Find the inverse of $h(x) = \dfrac{2x-5}{3}$.

Answer: $h^{-1}(x) = \dfrac{3x+5}{2}$.

Example 2. Verify the inverse from the previous example by composition.

Answer: $h(h^{-1}(x)) = x$ and $h^{-1}(h(x)) = x$.

Example 3. Use the table for f to make a table for f^{-1}.

Input x	-2	0	3	5
Output $f(x)$	7	1	-4	-6

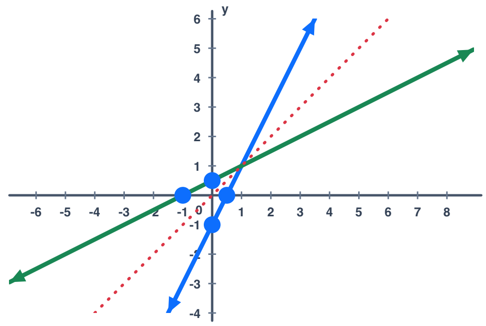
Answer: $f^{-1}(7) = -2$, $f^{-1}(1) = 0$, $f^{-1}(-4) = 3$, and $f^{-1}(-6) = 5$.

Input x	7	1	-4	-6
Output $f^{-1}(x)$	-2	0	3	5

Example 4. If f has domain $[-2, 6]$ and range $[1, 9]$, state the domain and range of f^{-1}.

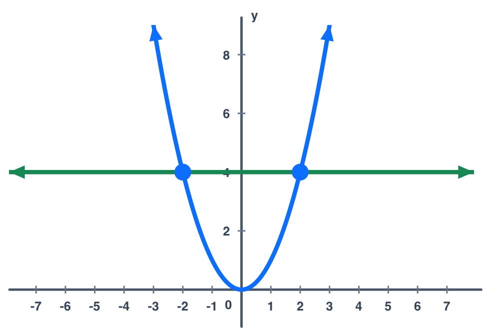
Answer: **Domain of f^{-1}: $[1, 9]$. Range of f^{-1}: $[-2, 6]$.**

Example 5. Use the graph of $f(x) = 2x - 1$, $f^{-1}(x) = \dfrac{x+1}{2}$, and $y = x$. Describe how the inverse graph is related to the original graph.



Answer: The inverse graph is the reflection of the original graph across $y = x$. Points swap coordinates: $(0, -1)$ on f corresponds to $(-1, 0)$ on f^{-1}. The red dotted line is the guide $y = x$, not one of the functions.

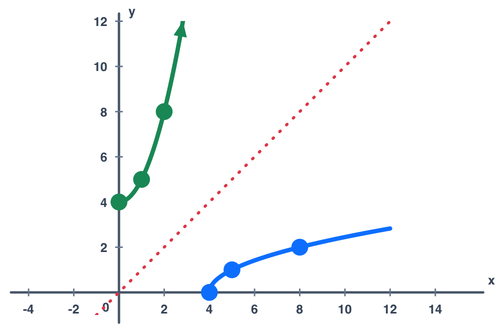
Example 6. Use the graph to explain why $f(x) = x^2$ does not have an inverse function on all real numbers. Give a domain restriction that works and state the resulting inverse.



Answer: The line $y = 4$ intersects the graph twice, so f fails the horizontal line test. Restricting the domain to $[0, \infty)$ gives $f^{-1}(x) = \sqrt{x}$.

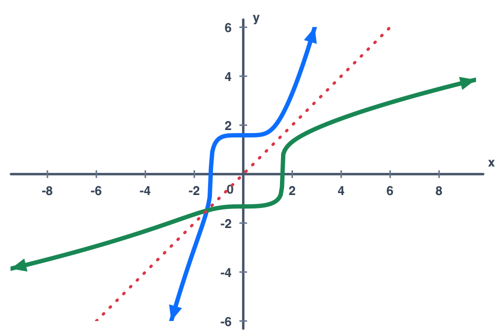
Example 7. For $f(x) = \sqrt{x-4}$, find $f^{-1}(x)$, then state the domain and range of f and f^{-1}. Compare the graph of f to the graph of its inverse.

Answer: $f^{-1}(x) = x^2 + 4$. **Domain of** f: $[4, \infty)$. **Range of** f: $[0, \infty)$. **Domain of** f^{-1}: $[0, \infty)$. **Range of** f^{-1}: $[4, \infty)$. **The blue graph of** f **and green graph of** f^{-1} **are reflections across the red dotted line** $y = x$.



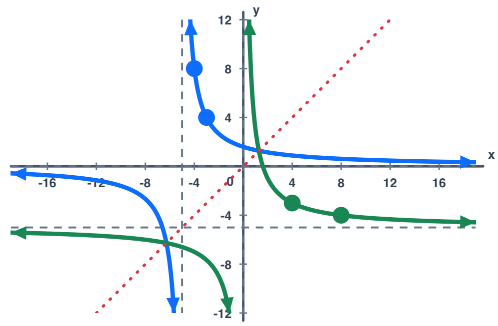
Example 8. For $f(x) = \sqrt[3]{x^5 + 4}$, find $f^{-1}(x)$, then state the domain and range of f and f^{-1}. Compare the graph of f to the graph of its inverse.

Answer: $f^{-1}(x) = \sqrt[5]{x^3 - 4}$. **The domain and range of both** f **and** f^{-1} **are all real numbers,** $(-\infty, \infty)$. **The blue graph of** f **and green graph of** f^{-1} **are reflections across the red dotted line** $y = x$.



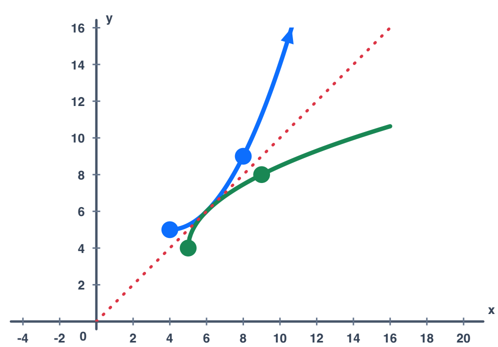
Example 9. For $f(x) = \dfrac{8}{5+x}$, find $f^{-1}(x)$, then state the domain and range of f and f^{-1}. Compare the graph of f to the graph of its inverse.

Answer: $f^{-1}(x) = \dfrac{8}{x} - 5$. **Domain of** f: $(-\infty, -5) \cup (-5, \infty)$. **Range of** f: $(-\infty, 0) \cup (0, \infty)$. **Domain of** f^{-1}: $(-\infty, 0) \cup (0, \infty)$. **Range of** f^{-1}: $(-\infty, -5) \cup (-5, \infty)$. **The blue graph of** f **and green graph of** f^{-1} **are reflections across the red dotted line** $y = x$.



Example 10. For $f(x) = \dfrac{1}{4}(x-4)^2 + 5$, $x \geq 4$, find $f^{-1}(x)$, then state the domain and range of f and f^{-1}. Compare the graph of f to the graph of its inverse.

Answer: $f^{-1}(x) = 4 + 2\sqrt{x-5}$. **Domain of** f: $[4, \infty)$. **Range of** f: $[5, \infty)$. **Domain of** f^{-1}: $[5, \infty)$. **Range of** f^{-1}: $[4, \infty)$. **The blue graph of** f **and green graph of** f^{-1} **are reflections across the red dotted line** $y = x$.



Module 8: Exponential and Logarithmic Functions

Students study exponential and logarithmic functions, connect them as inverses, solve equations, and build contextual models.

HW 34 Exponent Rules

Review exponent rules, including zero, negative, and rational exponents.

QUICK REFERENCE

- Use the exponent rule that matches the structure of the expression.
  - When multiplying powers with the same base, add the exponents.
  - When dividing powers with the same base, subtract the exponents.
  - When raising a power to another power, multiply the exponents. Apply an outside exponent to every factor inside parentheses.
  - A nonzero base raised to the zero power equals 1.
  - A negative exponent moves a factor across the fraction bar. It does not make the factor negative.
  - In $a^{m/n}$, the denominator n gives the root and the numerator m gives the power.
- Simplify numerical coefficients and variable factors separately, and write final answers using positive exponents unless told otherwise.

Example 1. Simplify $\dfrac{p^4p^7}{p^3}$.

Answer: p^8.

Example 2. Simplify $(2m^3n^2)^2$.

Answer: $4m^6n^4$.

Example 3. Simplify $\dfrac{5x^0y^{-3}}{z^{-2}}$ using positive exponents.

Answer: $\dfrac{5z^2}{y^3}$.

Example 4. Simplify $(64x^8y^3z^{12}) \div (20x^3yz^{15})$ using positive exponents.

Answer: $\dfrac{16x^5y^2}{5z^3}$.

Example 5. Rewrite $16^{3/4}$ in radical form and evaluate.

Answer: $16^{3/4} = (\sqrt[4]{16})^3 = 8$.

Example 6. Rewrite $\sqrt[3]{a^5}$ using a rational exponent.

Answer: $a^{5/3}$.

Example 7. Simplify $\left(27u^6v^3\right)^{1/3}$.

Answer: $3u^2v$.

Example 8. Simplify $\dfrac{(x^{-3}y^{1/2})^2}{xy^{-1}}$ using positive exponents.

Answer: $\dfrac{y^2}{x^7}$.

HW 35 Exponential Functions

Build growth, decay, transformation, and asymptote behavior for exponential functions.

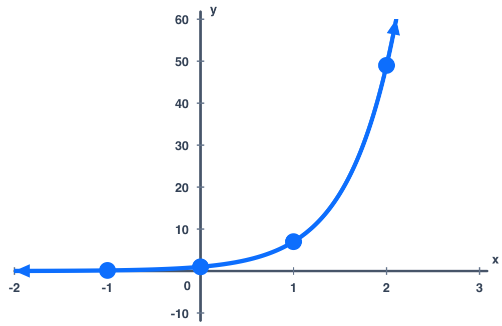
QUICK REFERENCE

- An exponential function has the variable in the exponent, often $A(t) = ab^t$.
- In $A(t) = ab^t$, the values of a and b describe the model.
  - The starting value is a, and the y-intercept is $(0, a)$.
  - The growth or decay factor is b. If $b > 1$, the model grows; if $0 < b < 1$, the model decays.
- The parent graph and transformed form show the important exponential features.
  - The parent function $f(x) = b^x$ has domain all real numbers, $(-\infty, \infty)$, range $(0, \infty)$, and horizontal asymptote $y = 0$.
  - For $f(x) = ab^{x-h} + k$, h shifts the graph left or right, k shifts it up or down, and the horizontal asymptote is $y = k$.
  - A negative value of a reflects the graph across the x-axis. Replacing x with $-x$ reflects it across the y-axis.
- Points on an exponential graph can reveal the base, especially when the y-values multiply by the same factor each step.
- The number e is the base of the natural exponential function. Calculator approximation is often used for powers such as $17^{3.14}$ or $e^{2.9}$.

Example 1. Approximate $17^{3.14}$, $(1 + 0.07)^8$, $e^{2.9}$, and $125e^{0.043(8)}$ to three decimals.

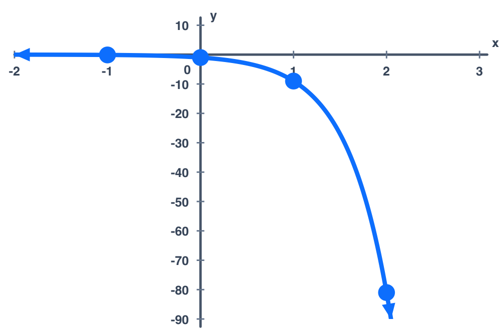
Answer: $17^{3.14} \approx 7304.822$. $(1 + 0.07)^8 \approx 1.718$. $e^{2.9} \approx 18.174$. $125e^{0.043(8)} \approx 176.322$.

Example 2. Use the graph and the points $\left(-1, \frac{1}{7}\right)$, $(0,1)$, $(1,7)$, and $(2,49)$ to determine the exponential function.



Answer: $f(x) = 7^x$.

Example 3. Use the graph and the points $\left(-1, -\frac{1}{9}\right)$, $(0,-1)$, $(1,-9)$, and $(2,-81)$ to determine the exponential function.



Answer: $f(x) = -9^x$.

Example 4. For $A(t) = 120(1.08)^t$, identify starting value, growth/decay factor, percent change, y-intercept, and horizontal asymptote.

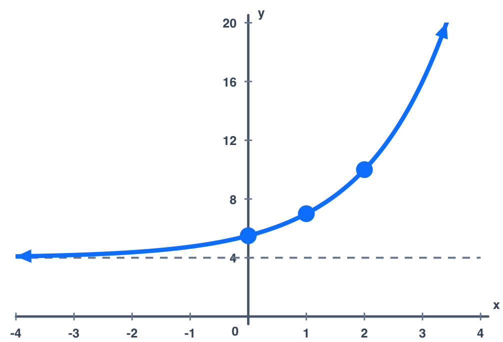
Answer: Starting value: 120. **Growth factor:** 1.08. **Increase:** 8%. **y-intercept:** $(0,120)$. **Horizontal asymptote:** $y = 0$.

Example 5. For $B(t) = 80(0.75)^t$, identify the starting value, growth/decay factor, percent change, y-intercept, and horizontal asymptote. Then find $B(3)$.

Answer: Starting value: 80. **Decay factor:** 0.75. **Decrease:** 25%. **y-intercept:** $(0,80)$. **Horizontal asymptote:** $y = 0$. $B(3) = 33.75$.

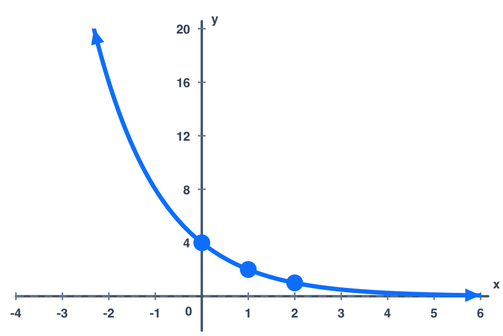
Example 6. Graph $f(x) = 3(2)^{x-1} + 4$. State the transformations from $y = 2^x$, domain, range, horizontal asymptote, and y-intercept.

Answer: Vertical stretch by 3; right 1; up 4. Domain: all real numbers, $(-\infty, \infty)$. Range: $(4, \infty)$. Horizontal asymptote: $y = 4$. y-intercept: $\left(0, \dfrac{11}{2}\right)$.



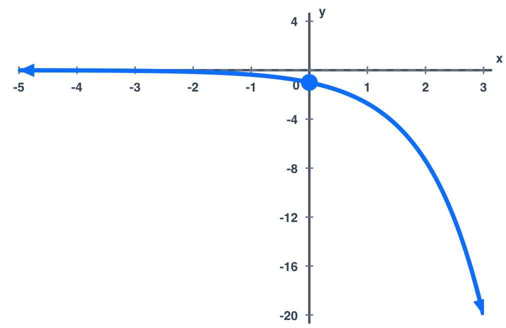
Example 7. Graph $g(x) = 4\left(\dfrac{1}{2}\right)^x$. State whether it shows growth or decay, domain, range, horizontal asymptote, and y-intercept.

Answer: Decay. Domain: all real numbers, $(-\infty, \infty)$. Range: $(0, \infty)$. Horizontal asymptote: $y = 0$. y-intercept: $(0, 4)$.



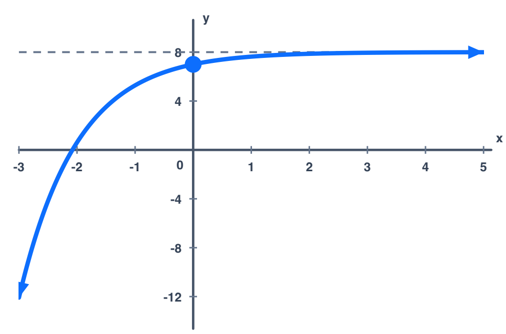
Example 8. Graph $h(x) = -e^x$. State the transformation from $y = e^x$, domain, range, horizontal asymptote, and y-intercept.

Answer: Reflection across the x-axis. Domain: all real numbers, $(-\infty, \infty)$. Range: $(-\infty, 0)$. Horizontal asymptote: $y = 0$. y-intercept: $(0, -1)$.



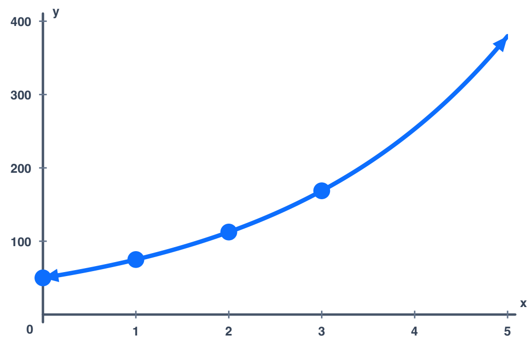
Example 9. Graph $f(x) = 8 - e^{-x}$. State the transformations from $y = e^x$, domain, range, horizontal asymptote, and y-intercept.

Answer: Reflection across the y-axis, reflection across the x-axis, and up 8. Domain: all real numbers, $(-\infty, \infty)$. Range: $(-\infty, 8)$. Horizontal asymptote: $y = 8$. y-intercept: $(0, 7)$.



Example 10. Use the table and graph of $A(t) = 50(1.5)^t$. Identify the starting value, growth factor, and percent increase.

t	0	1	2	3
$A(t)$	50	75	112.5	168.75



Answer: Starting value: 50. Growth factor: 1.5. Increase: 50% each time period.

HW 36 Logarithms and Logarithmic Functions

Convert forms, evaluate logarithms, and analyze logarithmic graphs and inverse relationships.

QUICK REFERENCE

- A logarithm answers the question "what exponent is needed?"
  - The statement $\log_b M = p$ means $b^p = M$, where $b > 0$, $b \neq 1$, and $M > 0$.
  - The notation $\log x$ means base 10.
  - The notation $\ln x$ means base e.
- Exact log values come from known powers. Use a calculator when the value is not based on a familiar power.
- The parent graph and transformed form show the important logarithmic features.
  - The parent function $f(x) = \log_b x$ has domain $(0, \infty)$, range all real numbers, $(-\infty, \infty)$, x-intercept $(1, 0)$, and vertical asymptote $x = 0$.
  - If $b > 1$, $f(x) = \log_b x$ is increasing. If $0 < b < 1$, it is decreasing.
  - For $f(x) = a\log_b(x - h) + k$, the graph shifts with h and k, and its vertical asymptote is $x = h$. The log input must stay positive.
- For logs of fractions, the entire argument must be positive; numerator and denominator signs both matter.
- Logarithmic and exponential functions are inverses, so their graphs reflect across $y = x$. Their domains and ranges switch.

Example 1. Rewrite each statement in logarithmic form: $2^{-3} = \frac{1}{8}$, $4.2 = a^4$, $6.9 = 8^x$, and $e^x = 6$.

Answer: $\log_2\left(\frac{1}{8}\right) = -3$. $\log_a(4.2) = 4$. $\log_8(6.9) = x$. $\ln 6 = x$.

Example 2. Rewrite each statement in exponential form: $\log_3 81 = 4$, $\log_a 5 = 8$, $\log_4 16 = x$, and $\ln 11 = x$.

Answer: $3^4 = 81$. $a^8 = 5$. $4^x = 16$. $e^x = 11$.

Example 3. Evaluate exactly without using a calculator: $\log_7 1$, $\log_5 125$, $\log_2\left(\frac{1}{16}\right)$, $\log_{1/3} 243$, and $\log_8 \sqrt{8}$.

Answer: 0, 3, -4, -5, and $\frac{1}{2}$.

Example 4. Use a calculator to evaluate $\ln\left(\frac{49}{47}\right)$ and $\frac{\ln(65/51)}{-0.04}$. Round to three decimals.

Answer: $\ln\left(\frac{49}{47}\right) \approx 0.042$. $\frac{\ln(65/51)}{-0.04} \approx -6.064$.

Example 5. Find the domain of $f(x) = \log(x+5)$.

Answer: $(-5, \infty)$.

Example 6. Find the domain of $h(x) = \ln\left(\frac{1}{x+8}\right)$.

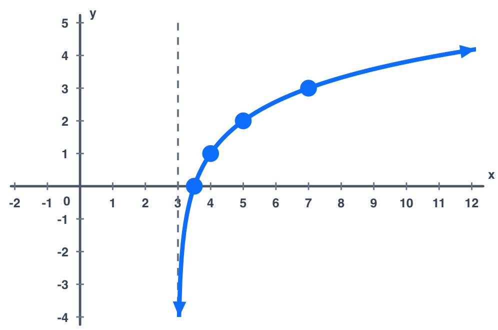
Answer: $(-8, \infty)$.

Example 7. Find the domain of $g(x) = \log_9\left(\frac{x+9}{x}\right)$.

Answer: $(-\infty, -9) \cup (0, \infty)$.

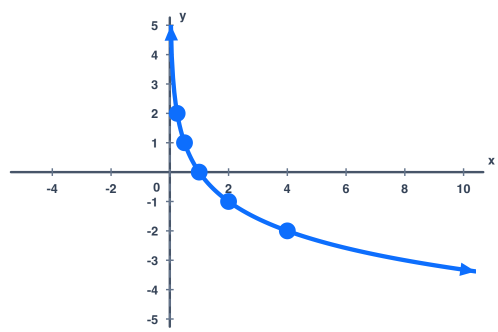
Example 8. Graph $L(x) = \log_2(x-3)+1$. State the transformations from $y = \log_2 x$, domain, range, vertical asymptote, and x-intercept.

Answer: Right 3; up 1. Domain: $(3, \infty)$. Range: all real numbers, $(-\infty, \infty)$. Vertical asymptote: $x = 3$. x-intercept: $\left(\frac{7}{2}, 0\right)$.



Example 9. Graph $g(x) = \log_{1/2} x$. State whether it is increasing or decreasing, domain, range, vertical asymptote, and x-intercept.

Answer: Decreasing. Domain: $(0, \infty)$. Range: all real numbers, $(-\infty, \infty)$. Vertical asymptote: $x = 0$. x-intercept: $(1, 0)$.



Example 10. Use the table to connect $y = 2^x$ and $y = \log_2 x$ as inverse functions.

Statement	Value 1	Value 2	Value 3	Value 4
Exponential	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$
Logarithmic	$\log_2 \frac{1}{2} = -1$	$\log_2 1 = 0$	$\log_2 2 = 1$	$\log_2 4 = 2$

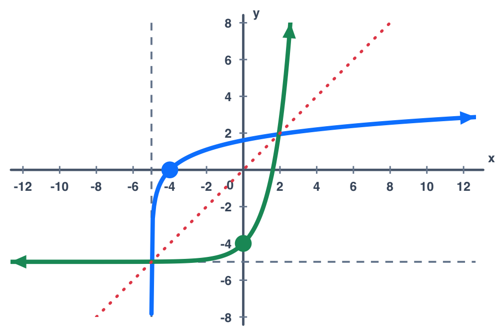
Answer: The input and output switch. For example, $2^2 = 4$ matches $\log_2 4 = 2$.

Example 11. For $f(x) = e^x + 5$, state domain, range, horizontal asymptote, inverse, and the domain, range, and vertical asymptote of the inverse.

> **Answer:** Domain of f: all real numbers, $(-\infty, \infty)$. Range of f: $(5, \infty)$. Horizontal asymptote: $y = 5$. $f^{-1}(x) = \ln(x-5)$. Domain of f^{-1}: $(5, \infty)$. Range of f^{-1}: all real numbers, $(-\infty, \infty)$. Vertical asymptote of f^{-1}: $x = 5$.

Example 12. For $f(x) = \ln(x+5)$, graph f and f^{-1} on the same axes. State the domain, range, and vertical asymptote of f; find $f^{-1}(x)$; and state the domain, range, and horizontal asymptote of the inverse.

> **Answer:** Domain of f: $(-5, \infty)$. Range of f: all real numbers, $(-\infty, \infty)$. Vertical asymptote of f: $x = -5$. $f^{-1}(x) = e^x - 5$. Domain of f^{-1}: all real numbers, $(-\infty, \infty)$. Range of f^{-1}: $(-5, \infty)$. Horizontal asymptote of f^{-1}: $y = -5$. The graphs reflect across $y = x$.



HW 37 Properties of Logarithms

Expand, condense, and change bases while respecting domain restrictions.

QUICK REFERENCE

- Log rules rewrite products, quotients, and powers. They do not split sums or differences inside a log.
- The inverse relationships are $\log_b(b^x) = x$ and $b^{\log_b M} = M$.
- Three main log rules handle products, quotients, and powers.
  - Product rule: $\log_b(MN) = \log_b M + \log_b N$.
  - Quotient rule: $\log_b\left(\dfrac{M}{N}\right) = \log_b M - \log_b N$.
  - Power rule: $\log_b(M^p) = p\log_b M$.
- Expanding turns products, quotients, and powers into sums, differences, and coefficients.
- Condensing reverses the log rules to write a single logarithm. Logarithms must have the same base before they can be combined.
- Change of base: $\log_b M = \dfrac{\ln M}{\ln b}$.
- When expanding or condensing, keep domain restrictions in mind: every log input must be positive.

Example 1. Evaluate exactly: $\log_8(8^{29})$, $\ln(e^{-11})$, $7^{\log_7 5}$, and $e^{\ln 12}$.

Answer: $29,\ -11,\ 5,\ 12$

Example 2. Evaluate exactly using log properties: $\log_{18} 6 + \log_{18} 3$ and $\log_6 30 - \log_6 5$.

Answer: $1,\ 1$

Example 3. Expand and simplify, assuming all log inputs are positive: $\log_5(125x)$, $\log_5(z^7)$, and $\ln(e(t+6))$.

Answer: $3 + \log_5 x,\ 7\log_5 z,\ 1 + \ln(t+6)$

Example 4. Expand and simplify: $\ln\left(\frac{t-7}{e^t}\right)$, where $t > 7$.

Answer: $\ln(t-7) - t$

Example 5. Expand completely: $\log_d(u^3v^8)$, where $u > 0$ and $v > 0$.

Answer: $3\log_d u + 8\log_d v$

Example 6. Expand completely: $\log_3\left(\frac{x^2\sqrt{y}}{z^5}\right)$, where $x > 0$, $y > 0$, and $z > 0$.

Answer: $2\log_3 x + \frac{1}{2}\log_3 y - 5\log_3 z$

Example 7. Expand completely: $\log_6\left(\frac{x^4}{x-4}\right)$, where $x > 4$.

Answer: $4\log_6 x - \log_6(x-4)$

Example 8. Explain why $\log(x+3) \neq \log x + \log 3$.

Answer: **The product rule works for multiplication inside the log, not addition.**

Example 9. State the domain restriction for $\log(x-2) + \log(x+5)$.

Answer: $x > 2$

Example 10. Condense: $2\log_3 u + 6\log_3 v$, where $u > 0$ and $v > 0$.

Answer: $\log_3(u^2v^6)$

Example 11. Condense: $2\ln x - \frac{1}{3}\ln y + \ln 4$, where $x > 0$ and $y > 0$.

Answer: $\ln\left(\frac{4x^2}{\sqrt[3]{y}}\right)$

Example 12. Condense: $6\log_2\sqrt{3x-1}-\log_2\left(\dfrac{8}{x}\right)+\log_2 8$. State the domain restriction.

Answer: $\log_2\left(x(3x-1)^3\right)$, with $x>\dfrac{1}{3}$

Example 13. Use change of base to approximate $\log_4 28$, $\log_{1/6} 2$, and $\log_{\sqrt{13}} 12$ to three decimals.

Answer: $\dfrac{\ln 28}{\ln 4}\approx 2.404,\quad \dfrac{\ln 2}{\ln(1/6)}\approx -0.387,\quad \dfrac{\ln 12}{\ln(\sqrt{13})}\approx 1.938$

Example 14. Use numerical values to check the product rule $\log_2(4\cdot 8)=\log_2 4+\log_2 8$.

Expression	$\log_2(4\cdot 8)$	$\log_2 4+\log_2 8$
Value	5	$2+3=5$

Answer: Both sides equal 5.

HW 38 Exponential and Logarithmic Equations

Solve exponential and logarithmic equations using inverse relationships, log properties, and domain checks.

QUICK REFERENCE

- Choose an exponential-equation method based on the form of the equation.
  - If the expressions can be written with the same base, match their exponents.
  - When the bases do not match, isolate the exponential expression and take a logarithm of both sides.
  - Some equations become quadratic after substituting $u=b^x$. Since $b^x>0$, only positive values of u can be used.
  - An exact logarithmic answer can be written in compact form, such as $\log_b M$, or with change of base, $\dfrac{\ln M}{\ln b}$.
- Choose a logarithmic-equation method based on the form of the equation.
  - For an isolated logarithm, rewrite the equation in exponential form.
  - If $\log_b M=\log_b N$, then $M=N$, as long as both log inputs are positive.
  - Use log properties to condense an equation before rewriting it in exponential form.
  - Every solution must make each original log input positive. A domain check can remove a candidate or show that there is no solution.
- A graph or table can help estimate and check a solution.

Example 1. Solve $2^{x+1}=16$.

Answer: $x=3$

Example 2. Solve $2^{x^2} = 128^x$.

Answer: $x = 0, 7$

Example 3. Solve $8^x = 3$. Give the exact solution and a decimal approximation to three places.

Answer: Exact solution: $x = \log_8 3 = \dfrac{\ln 3}{\ln 8}$. Decimal approximation: $x \approx 0.528$.

Example 4. Solve $7(5^{5t}) = 4$ exactly.

Answer: $t = \dfrac{1}{5}\log_5\left(\dfrac{4}{7}\right) = \dfrac{\ln(4/7)}{5\ln 5}$

Example 5. Solve $4^{1-8x} = 7^x$ exactly.

Answer: $x = \dfrac{1}{8 + \log_4 7} = \dfrac{\ln 4}{8\ln 4 + \ln 7}$

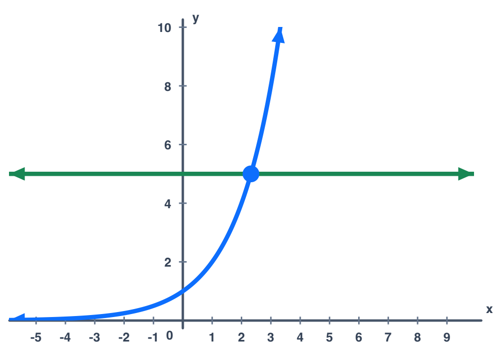
Example 6. Solve $5^{2x} + 5^x - 6 = 0$ exactly.

Answer: $x = \log_5 2 = \dfrac{\ln 2}{\ln 5}$

Example 7. Solve $4^w + 2^{w+1} - 1 = 0$ exactly.

Answer: $w = \log_2(\sqrt{2} - 1) = \dfrac{\ln(\sqrt{2} - 1)}{\ln 2}$

Example 8. Use the graph to estimate the solution of $2^x = 5$, then write the exact logarithmic solution.



Answer: Estimate: $2 < x < 2.5$. Exact solution: $x = \log_2 5 = \dfrac{\ln 5}{\ln 2}$.

Example 9. Solve each logarithmic equation: $\log_3 x = 2$ and $\log_7(11x) = 2$.

Answer: $x = 9$ and $x = \dfrac{49}{11}$

Example 10. Solve $\log_9(x + 7) = \log_9 11$.

Answer: $x = 4$

Example 11. Solve $\frac{1}{2}\log_9 x = 2\log_9 5$.

Answer: $x = 625$

Example 12. Solve $2\log_9(x-5) + \log_9 9 = 2$.

Answer: $x = 8$

Example 13. Solve $\log x + \log(x-48) = 2$, and check the domain.

Answer: $x = 50$. **Excluded candidate:** $x = -2$.

Example 14. Solve $\log(x-2) = \log(1-x)$, and check the domain.

Answer: **No solution,** $\varnothing$.

Example 15. Solve $\ln x + \ln(x+6) = 3$. Give the exact solution and a decimal approximation to three places.

Answer: **Exact solution:** $x = -3 + \sqrt{9+e^3}$. **Decimal approximation:** $x \approx 2.393$.

Example 16. Solve $2\log(x+5) = \log(10x+39)$.

Answer: $x = -\sqrt{14}, \sqrt{14}$

Example 17. Solve $\log_{1/7}(x^2+x) - \log_{1/7}(x^2-x) = -1$.

Answer: $x = \frac{4}{3}$

HW 39 Exponential and Logarithmic Modeling

Model compound interest, present value, doubling time, half-life, growth, decay, and data interpretation.

QUICK REFERENCE

- Exponential models describe repeated percent change, growth, decay, interest, and half-life.
- The initial value is the amount at time 0. The growth or decay factor controls repeated change.

- Choose the exponential model that matches the application.
  - Periodic compound interest uses $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where P is the principal, r is the annual rate, n is the number of compoundings per year, and t is time in years.
  - Use $n = 1$ for annual, 4 for quarterly, 12 for monthly, and 365 for daily compounding.
  - Continuous compounding uses $A = Pe^{rt}$.
  - Continuous growth and decay use $A(t) = A_0 e^{kt}$. In $A_0 e^{kt}$, k is the continuous growth or decay constant and e^k is the factor for one time period.
  - Doubling-time and half-life models use powers such as $2^{t/d}$ or $\left(\frac{1}{2}\right)^{t/h}$.
  - Present value means solving for the starting amount P. Finding time usually requires logarithms.
- Regression models should match the pattern in the data and be interpreted carefully.
  - Exponential regression fits repeated percent change.
  - Logarithmic regression can model relationships that rise or fall quickly and then level off.
  - Use units and be cautious when predicting outside the data range.
- Modeling questions often ask for a formula, prediction, target time, or parameter interpretation. State units and follow the requested rounding.

Example 1. A $\$1200$ investment earns 5% interest compounded monthly for 6 years. Find the value to the nearest cent.

Answer: $\$1,618.82$

Example 2. A $\$2500$ investment earns 4.6% interest compounded continuously for 7 years. Find the value to the nearest cent.

Answer: $\$3,449.71$

Example 3. How much should be invested now at 4% compounded quarterly to have $\$5000$ in 8 years? Round to the nearest cent.

Answer: $\$3,636.52$

Example 4. How much should be invested now at 5% compounded continuously to have $\$10,000$ in 3 years? Round to the nearest cent.

Answer: $\$8,607.08$

Example 5. An investment grows from $\$1000$ to $\$1400$ at 5% annual interest. Find the time with quarterly compounding and with continuous compounding. Round each time to the nearest hundredth.

Answer: Quarterly compounding: 6.77 years. Continuous compounding: 6.73 years.

Example 6. An insect population follows $P(t) = 500e^{0.08t}$, where t is measured in days. Find the initial population, continuous growth rate, population after 10 days, time to reach 700, and doubling time. Round populations to the nearest whole number and times to the nearest tenth.

Answer: **Initial population: 500 insects. Continuous growth rate: 8% per day. $P(10) \approx 1113$ insects. The population reaches 700 in about 4.2 days and doubles in about 8.7 days.**

Example 7. A town has an initial population of $12{,}000$ and doubles every 15 years. Write a model and predict the population after 40 years to the nearest whole number.

Answer: $P(t) = 12000(2)^{t/15}$. $P(40) \approx 76{,}195$ **people.**

Example 8. A radioactive sample follows $A(t) = 500e^{-0.024t}$, where t is measured in years. Find the continuous decay rate, amount after 40 years, time until 300 grams remain, and half-life. Round the amount and times to the nearest tenth.

Answer: **Continuous decay rate: -2.4% per year. $A(40) \approx 191.4$ grams. 300 grams remain after about 21.3 years. Half-life: about 28.9 years.**

Example 9. A substance has a half-life of 9 hours. If there are initially 60 grams, write a model and find the amount after 27 hours.

Answer: $A(t) = 60\left(\frac{1}{2}\right)^{t/9}$. $A(27) = 7.5$ **grams.**

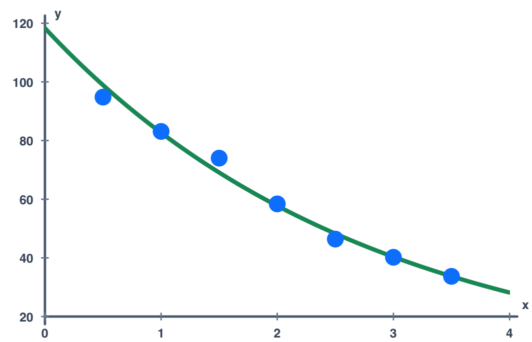
Example 10. Carbon-14 has a half-life of 5730 years. A sample contains 68% of its original carbon-14. Estimate the sample's age to the nearest whole year.

Answer: **Approximately 3188 years old.**

Example 11. Use exponential regression on the survival data. Write the model in the forms $S(t) = ab^t$ and $S(t) = Ae^{kt}$, rounding model values to four decimals. Graph the model with the data, predict $S(3)$ to the nearest tenth of a percent, and interpret b and k.

Years after diagnosis	0.5	1	1.5	2	2.5	3	3.5
Percent surviving	94.8	83.1	74.0	58.4	46.4	40.2	33.7

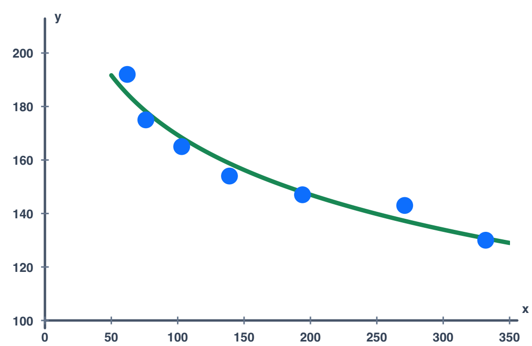
Answer: $S(t) \approx 118.3208(0.6986)^t = 118.3208e^{-0.3587t}$. $S(3) \approx 40.3\%$. **The model retains about 69.86% each year, a decrease of about 30.14%. The equivalent continuous decay constant is $k \approx -0.3587$.**



Example 12. Use logarithmic regression with dairy farms f, in thousands, as the input and milk production M, in billions of pounds, as the output. Find $M(f) = a + b\ln(f)$ to four decimals and graph the model with the data. Then predict production when $f = 65$. If the actual production was 190 billion pounds, compare the prediction with the actual value.

Year	Dairy farms (thousands)	Milk produced (billion pounds)
1980	332	130
1985	271	143
1990	194	147
1995	139	154
2000	103	165
2005	76	175
2010	62	192

Answer: $M(f) \approx 317.6949 - 32.2131\ln(f)$. $M(65) \approx 183.2$ **billion pounds. The prediction is about** 6.8 **billion pounds less than the actual value.**



Example 13. Use the table for $A(t) = 500(1.12)^t$. Estimate when the amount first exceeds 800.

t years	0	1	2	3	4	5
$A(t)$	500	560	627.20	702.46	786.76	881.17

Answer: **From the table, the amount first exceeds** 800 **between** 4 **and** 5 **years. At the listed whole-year values, it first exceeds** 800 **at** $t = 5$.

Module 9: Systems of Equations and Decomposition

Students solve systems analytically, graphically, and numerically, then use systems in applications and partial fraction decomposition.

HW 40 Systems by Graphing and Tables

Interpret intersections, verify ordered pairs, and classify system behavior.

QUICK REFERENCE

- A system asks where two or more equations are true at the same time.
- To verify a listed ordered pair, substitute it into every equation in the system.
- When several ordered pairs are listed, more than one pair may work, or none may work.
- Graphically, a solution is an intersection point. Tables show a solution where the two outputs match for the same input.
- Rewrite an equation in slope-intercept form, $y = mx + b$, when that makes it easier to graph. A vertical line stays in the form $x = a$.
- The graphs show which of the three system outcomes occurs.
  - One intersection means one solution. This is a consistent and independent system.
  - Parallel lines mean no solution, $\varnothing$. This is an inconsistent system.
  - The same line means infinitely many solutions. This is a consistent and dependent system, and the two graphs may look like one line.
- A solution to a two-variable system is written as an ordered pair (x, y).

Example 1. Verify whether $(3, -1)$ is a solution to $3x - y = 10$ and $6x + 5y = 13$.

Answer: Yes. $(3, -1)$ is a solution.

Example 2. Verify whether $(9, 3)$ is a solution to $y = \frac{1}{3}x$ and $x - 2y = 3$.

Answer: Yes. $(9, 3)$ is a solution.

Example 3. Which ordered pairs satisfy $x + y = 4$ and $2x - y = 2$: $(0, 4)$, $(2, 2)$, or $(3, 1)$?

Answer: Only $(2, 2)$ works in both equations.

Example 4. Which ordered pairs satisfy $x + y = 5$ and $2x - y = 1$: $(0, 3)$, $(3, 1)$, or $(4, 0)$?

Answer: None of the listed ordered pairs works in both equations.

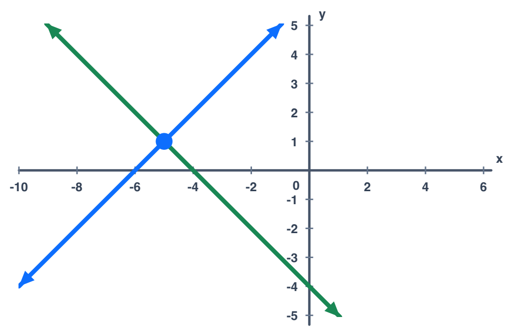
Example 5. Use the table to solve the system $y = 2x + 1$ and $y = -x + 7$.

x	1	2	3
$2x+1$	3	5	7
$-x+7$	6	5	4

Answer: $(2, 5)$.

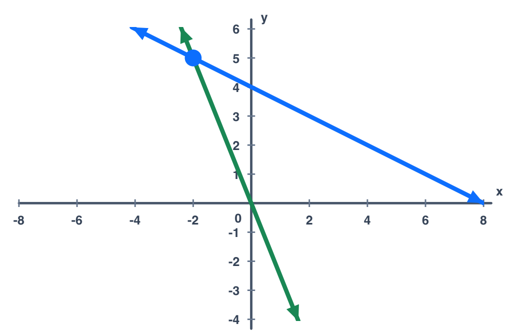
Example 6. Graph the system $y = x + 6$ and $y = -x - 4$. State the solution and classify the system.

Answer: $(-5, 1)$; consistent and independent.



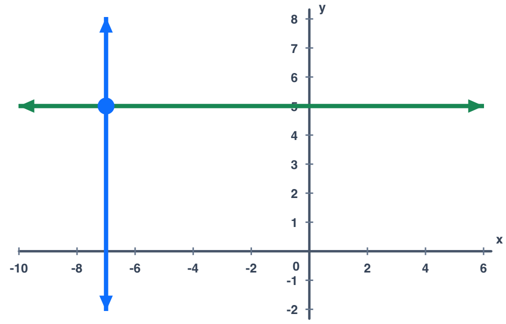
Example 7. Graph the system $x + 2y = 8$ and $5x + 2y = 0$. State the solution and classify the system.

Answer: $(-2, 5)$; consistent and independent.



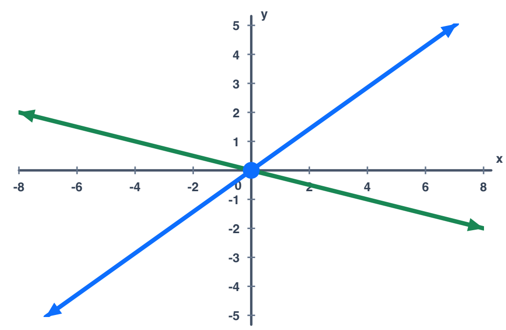
Example 8. Graph the system $-3x = 21$ and $y = 5$. State the solution and classify the system.

Answer: $(-7, 5)$**; consistent and independent.**



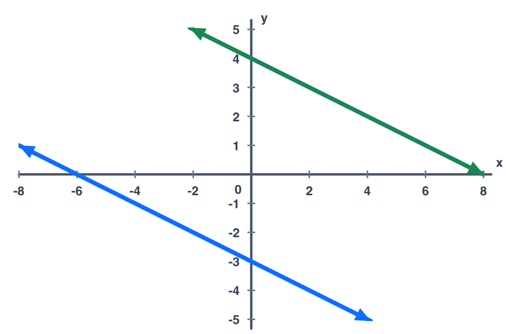
Example 9. Graph the system $5x - 7y = 0$ and $x + 4y = 0$. State the solution and classify the system.

Answer: $(0, 0)$**; consistent and independent.**



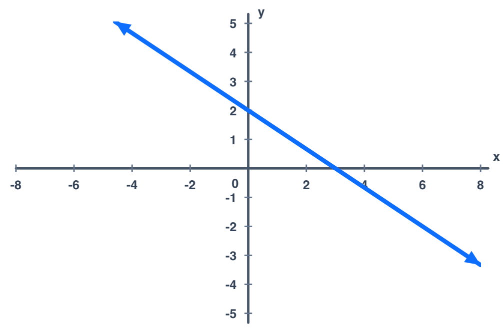
Example 10. Graph the system $y = -\dfrac{1}{2}x - 3$ and $x + 2y = 8$. State the solution and classify the system.

Answer: **No solution,** $\varnothing$**; inconsistent.**



Example 11. Graph the system $2x + 3y = 6$ and $4x + 6y = 12$. State the solution and classify the system.

Answer: **Infinitely many solutions; consistent and dependent.**



HW 41 Systems by Substitution and Elimination

Use substitution and elimination for exact algebraic solving.

QUICK REFERENCE

- Choose substitution or elimination based on the form of the system.
  - Substitution replaces one variable expression with another expression equal to it. It works best when a variable is already alone or is easy to get alone.
  - Elimination, sometimes called the addition method, adds equations to make one variable disappear.
  - If the coefficients do not cancel, multiply one or both equations first.
  - Fractions or decimals can be cleared first when that makes the system easier to solve.
- When both variables disappear, the remaining statement identifies the outcome.
  - A false statement, such as $0 = 5$, means no solution, $\varnothing$.
  - A true statement, such as $0 = 0$, means infinitely many solutions.
- A two-variable solution is written as an ordered pair (x, y) and should make both equations true.
- Three-variable systems extend the same ideas to ordered triples (x, y, z).
  - Eliminate the same variable from two different pairs of equations.
  - Solve the resulting two-variable system and substitute back.
  - The system can have one solution, no solution, or infinitely many solutions. When one variable is free, write the other variables in terms of it.

Example 1. Solve by substitution: $y = 2x - 3$ and $x + y = 12$.

Answer: $(5, 7)$.

Example 2. Solve by substitution: $x - 3y = -5$ and $4x + 2y = 22$.

Answer: $(4, 3)$.

Example 3. Solve by substitution: $3x - y = 1$ and $7x + \frac{1}{3}y = 1$.

Answer: $\left(\frac{1}{6}, -\frac{1}{2}\right)$.

Example 4. Solve by substitution and classify the system: $3x + y = 2$ and $6x + 2y = 9$.

Answer: No solution, $\varnothing$; inconsistent.

Example 5. Solve by substitution, classify the system, and write the solutions with y in terms of x: $y = \frac{1}{4}x - 1$ and $x - 4y = 4$.

Answer: Infinitely many solutions; consistent and dependent.
$\left\{(x, y) \mid y = \frac{1}{4}x - 1,\ x \text{ is any real number}\right\}$.

Example 6. Solve by elimination: $2x + 3y = 12$ and $4x - 3y = 6$.

Answer: $(3, 2)$.

Example 7. Solve by elimination: $x + 2y = 8$ and $3x - y = 3$.

Answer: $(2, 3)$.

Example 8. Solve by elimination: $5x - 4y = -11$ and $4x + 3y = -15$.

Answer: $(-3, -1)$.

Example 9. Solve by elimination: $6x - 9y = 7.5$ and $-2x + 7y = -4.5$.

Answer: $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

Example 10. Solve by elimination and classify the system: $2x + 3y = 8$ and $4x + 6y = 20$.

Answer: No solution, $\varnothing$; inconsistent.

Example 11. Solve by elimination, classify the system, and write the solutions with y in terms of x: $2x + 3y = 8$ and $4x + 6y = 16$.

Answer: Infinitely many solutions; consistent and dependent.
$\left\{(x, y) \mid y = -\frac{2}{3}x + \frac{8}{3},\ x \text{ is any real number}\right\}$.

Example 12. Use the check table to verify the solution $(3, 2)$ for $2x + 3y = 12$ and $4x - 3y = 6$.

Equation	Substitute $(3, 2)$	True?
$2x + 3y = 12$	$2(3) + 3(2) = 12$	Yes
$4x - 3y = 6$	$4(3) - 3(2) = 6$	Yes

Answer: **Yes. $(3, 2)$ makes both equations true.**

Example 13. Solve the three-variable system $x - y = 6$, $5x - 2z = 40$, and $5y + z = 10$.

Answer: $(8, 2, 0)$.

Example 14. Solve the three-variable system $x - 3y + 4z = 17$, $2x + y + z = 6$, and $-2x + 3y - 3z = -18$.

Answer: $(3, -2, 2)$.

Example 15. Classify the three-variable system $x - y - z = 2$, $5x + 3y + z = 3$, and $24x + 8y = 0$.

Answer: **No solution, $\varnothing$; inconsistent.**

Example 16. Solve and classify the three-variable system $x - y - z = 1$, $-x + 2y - 2z = -5$, and $4x - y - 13z = -8$. Write the solution set.

Answer: **Infinitely many solutions; consistent and dependent.** $\{(x, y, z) \mid x = 4z - 3,\ y = 3z - 4,\ z \text{ is any real number}\}$.

HW 42 Systems Applications

Translate applied settings into systems and interpret the results.

QUICK REFERENCE

- Keep the modeling process connected to the context.
  - Define variables with units before writing the system.
  - Use the facts in the problem to write one equation at a time. Total and comparison statements often provide the two equations you need.
  - Interpret the solution in a sentence with units.
  - Reject answers that do not make sense in context, such as negative ticket counts or impossible percentages.

- Common applications use familiar formulas and equation patterns.
  - For a rectangle, $P = 2L + 2W$. A second equation describes how the length and width are related.
  - Value problems use $\text{number} \times \text{price}$. Mixture problems use $\text{amount} \times \text{concentration}$.
  - Motion problems use $d = rt$. Add wind or current when it helps the motion and subtract it when it works against the motion.
  - Break-even is where two models have the same input and output. Write the answer as an ordered pair and interpret both coordinates.
  - Three-variable applications need three different equations, such as one equation for a total and two equations for revenue.
- Some problems require solving for unit prices before using those prices to answer the final question.

Example 1. A rectangular room has perimeter 76 feet. Its length is 8 feet more than its width. Define variables, write a system, and find the dimensions.

Answer: Let L be the length and W be the width, in feet. $2L + 2W = 76$ and $L = W + 8$. The room is 23 feet by 15 feet.

Example 2. A total of 68 commercial and noncommercial launches occurred in one year. The number of noncommercial launches was 8 more than twice the number of commercial launches. Define variables, write a system, and find both numbers.

Answer: Let c be commercial launches and n be noncommercial launches. $c + n = 68$ and $n = 2c + 8$. There were 20 commercial and 48 noncommercial launches.

Example 3. A restaurant will purchase 180 sets of dishes using its entire $\$5040$ budget. One design costs $\$24$ per set and another costs $\$36$ per set. Use the setup table to write and solve a system.

Dish design	Number of sets	Cost per set	Total cost
Design A	x	$\$24$	$24x$
Design B	y	$\$36$	$36y$
Total	180		$\$5040$

Answer: $x + y = 180$ and $24x + 36y = 5040$. The restaurant should purchase 120 sets of Design A and 60 sets of Design B.

Example 4. A theater sold 120 tickets. Adult tickets cost $\$12$, student tickets cost $\$8$, and total revenue was $\$1240$. Define variables, write a system, and find how many of each ticket were sold.

Answer: Let a be adult tickets and s be student tickets. $a + s = 120$ and $12a + 8s = 1240$. The theater sold 70 adult tickets and 50 student tickets.

Example 5. How many liters of 20% solution and 50% solution are needed to make 30 liters of a 32% solution? Define variables, write a system, and solve.

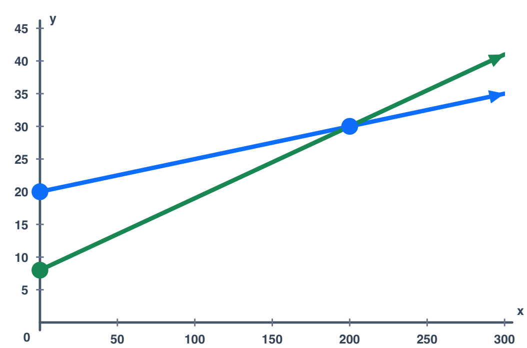
Answer: Let x and y be the amounts, in liters, of the 20% and 50% solutions. $x + y = 30$ and $0.20x + 0.50y = 9.6$. Use 18 liters of the 20% solution and 12 liters of the 50% solution.

Example 6. A store received 180 smartphone preorders and 120 tablet preorders totaling $\$165{,}000$. A smartphone costs $\$325$ more than a tablet. Define variables, write a system, and find both prices.

Answer: **Let s be the smartphone price and t be the tablet price. $180s + 120t = 165{,}000$ and $s = t + 325$. A smartphone costs $\$680$, and a tablet costs $\$355$.**

Example 7. Plan A costs $C_A(p) = 20 + 0.05p$ dollars, and Plan B costs $C_B(p) = 8 + 0.11p$ dollars, where p is the number of pages. Write a system, find and interpret the break-even point, and graph both models.

Answer: **$C = 20 + 0.05p$ and $C = 8 + 0.11p$. The break-even point is $(200, 30)$: both plans cost $\$30$ at 200 pages.**



Example 8. With a tailwind, an aircraft travels 480 miles in 2 hours. Against the same wind, it travels 480 miles in 3 hours. Define variables, write a system, and find the aircraft speed in still air and the wind speed.

Answer: **Let a be the aircraft speed in still air and w be the wind speed, in miles per hour. $a + w = 240$ and $a - w = 160$. The aircraft speed is 200 mph, and the wind speed is 40 mph.**

Example 9. One shopper buys 3 packages of bacon and 4 cartons of eggs for $\$24.85$. Another buys 2 packages of bacon and 5 cartons of eggs for $\$22.75$. Define variables, write and solve a system for the unit prices, and find the refund for returning 4 packages of bacon and 2 cartons of eggs.

Answer: **Let b be the price of one package of bacon and e be the price of one carton of eggs. $3b + 4e = 24.85$ and $2b + 5e = 22.75$. The unit prices are $b = \$4.75$ and $e = \$2.65$. The refund is $\$24.30$.**

Example 10. A theater has 360 seats divided among orchestra, main, and balcony sections. Tickets cost $\$80$, $\$55$, and $\$35$, respectively. A sold-out performance brings in $\$18{,}500$. Another performance sells all main and balcony seats but only half the orchestra seats and brings in $\$16{,}100$. Define variables, write a three-variable system, and find the number of seats in each section.

Answer: **Let o, m, and b be the orchestra, main, and balcony seat counts. $o + m + b = 360$, $80o + 55m + 35b = 18{,}500$, and $40o + 55m + 35b = 16{,}100$. There are 60 orchestra seats, 160 main seats, and 140 balcony seats.**

HW 43 Partial Fraction Decomposition

Use systems work to support partial fraction decomposition.

QUICK REFERENCE

- Partial fractions rewrite one rational expression as a sum of simpler rational expressions.
- Prepare the rational expression before setting up the decomposition.
  - The expression must be proper, meaning the numerator degree is less than the denominator degree. If it is improper, divide first.
  - Factor the denominator completely.
- The denominator factors determine the partial-fraction numerators.
  - For distinct linear factors, use one constant numerator over each factor.
  - For repeated linear factors, include a term for each power of the repeated factor.
  - For an irreducible quadratic factor, use a linear numerator such as $Bx + C$.
- After setting up the decomposition, solve for the unknown constants and check the result.
  - Multiply by the least common denominator to clear the fractions.
  - Choose convenient values that make factors zero and cause terms to disappear.
  - Find any remaining constants by comparing coefficients or solving a system.
  - Check by recombining the partial fractions and simplifying.

Example 1. Find the partial fraction decomposition of $\dfrac{5}{x(x-5)}$.

Answer: $\dfrac{5}{x(x-5)} = -\dfrac{1}{x} + \dfrac{1}{x-5}$.

Example 2. Find the partial fraction decomposition of $\dfrac{x}{(x-3)(x+5)}$.

Answer: $\dfrac{x}{(x-3)(x+5)} = \dfrac{3}{8(x-3)} + \dfrac{5}{8(x+5)}$.

Example 3. Set up the partial fraction form for $\dfrac{3x+7}{x(x-1)^2}$, but do not solve.

Answer: $\dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.

Example 4. Find the partial fraction decomposition of $\dfrac{x^2}{(x-1)^2(x+2)}$.

Answer: $\dfrac{x^2}{(x-1)^2(x+2)} = \dfrac{5}{9(x-1)} + \dfrac{1}{3(x-1)^2} + \dfrac{4}{9(x+2)}$.

Example 5. Find the partial fraction decomposition of $\dfrac{8}{x(x^2+4)}$.

Answer: $\dfrac{8}{x(x^2+4)} = \dfrac{2}{x} - \dfrac{2x}{x^2+4}$.

Example 6. Factor the denominator and find the partial fraction decomposition of $\dfrac{2}{x^3-1}$.

Answer: $x^3-1=(x-1)(x^2+x+1)$, and $\dfrac{2}{x^3-1}=\dfrac{2}{3(x-1)}-\dfrac{2x+4}{3(x^2+x+1)}$.

Example 7. Divide first, then rewrite $\dfrac{x^2+3x+1}{x+1}$.

Answer: $\dfrac{x^2+3x+1}{x+1}=x+2-\dfrac{1}{x+1}$.

Example 8. After clearing denominators, $4x+9=A(x+2)+B(x-1)$. Write and solve the coefficient system for A and B.

Answer: $A+B=4$ and $2A-B=9$. $A=\dfrac{13}{3}$ and $B=-\dfrac{1}{3}$.